Effective Action of Bosonic String Theory

Mehdi Ameri

Based on: arxiv.2506.19471

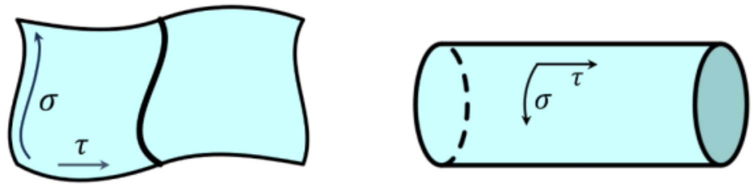
Co-Authors:
Alireza Pahlavan, Mohammad R. Garousi

September 2025

دانشگاه فردوسی مشهد

Introduction: Effective action

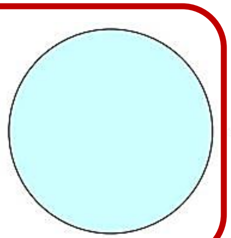
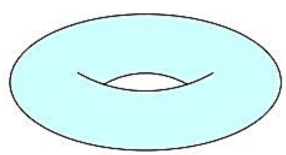
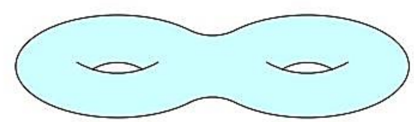
$$S_P = -\frac{T}{2}\int d^2\sigma\sqrt{-h}\ \left[h^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X) + \epsilon^{ab} B_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu + \alpha' R^{(2)}\Phi(X)\right]$$



World-Sheet (Open String) World-Sheet (Closed String)

$$X^\mu(\tau,\sigma)_R = \frac{1}{2}x^\mu + l_s^2 p^\mu(\tau-\sigma) + \text{oscillation part}$$
$$X^\mu(\tau,\sigma)_L = \frac{1}{2}x^\mu + l_s^2 p^\mu(\tau+\sigma) + \text{oscillation part}$$

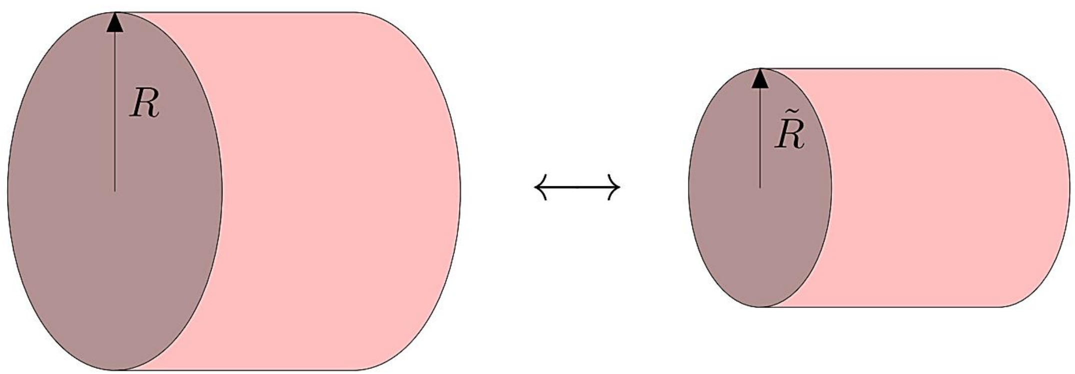
$$\alpha' M^2 = 4(N-1) = 4(\tilde{N}-1)$$

g_s^{-2}  $+\ g_s^{0}$  $+\ g_s^{2}$  $+ \dots$

A. Sen, Phys. Lett. B **271**, 295-300 (1991) doi:10.1016/0370-2693(91)90090-D.

T-duality

$$X^{\mu}(\sigma+\pi,\tau) = X^{\mu}(\sigma,\tau), \qquad \mu = 0,\ldots,24$$
$$X^{25}(\sigma+\pi,\tau) = X^{25}(\sigma,\tau) + 2\pi RW, \qquad W \in \mathbb{Z}$$
$$X^{25}(\sigma,\tau) = x^{25} + 2\alpha' P^{25}\tau + 2RW\sigma + \text{oscillation part}$$



$$X = x + \alpha' \frac{n}{R}\tau + mR\sigma + \text{modes}$$
$$\widetilde{X} = \tilde{x} + mR\tau + \alpha' \frac{n}{R}\sigma + \text{modes}$$

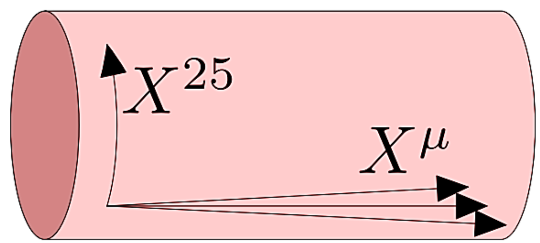
$$R \leftrightarrow \frac{\alpha'}{R} \qquad m \leftrightarrow n$$

T-duality in Target Space

Compactifying 26-dimensional space on a **circle** with radius R and choosing the background to be:

$$G_{\mu\nu} = \begin{pmatrix} \bar{g}_{ab} + e^{\varphi} g_a g_b & e^{\varphi} g_a \\ e^{\varphi} g_b & e^{\varphi} \end{pmatrix}, \quad G^{\mu\nu} = \begin{pmatrix} \bar{g}^{ab} & -g^a \\ -g^b & e^{-\varphi} + g_c g^c \end{pmatrix}$$

$$B_{\mu\nu} = \begin{pmatrix} \bar{b}_{ab} + \frac{1}{2} b_a g_b - \frac{1}{2} b_b g_a & b_a \\ -b_b & 0 \end{pmatrix}, \quad \Phi = \bar{\phi} + \varphi/4\,,$$



*Buscher rules (in target space):

$$\boxed{\varphi' = -\varphi \;,\; g'_a = b_a \;,\; b'_a = g_a \;,\; \bar{g}'_{ab} = \bar{g}_{ab} \;,\; \bar{b}'_{ab} = \bar{b}_{ab} \;,\; \bar{\phi}' = \bar{\phi}\,,}$$

T. H. Buscher, “Phys. Lett. B **194**, 59 (1987). doi:10.1016/0370-2693(87)90769-6

Circular Reduction

$$\begin{aligned}
e^{-2\Phi}\sqrt{-G} &= e^{-2\bar{\phi}}\sqrt{-\bar{g}} \\
R &= -\nabla^a\nabla_a\varphi - \frac{1}{2}\nabla_a\varphi\nabla^a\varphi - \frac{1}{4}e^{\varphi}V^2 \\
R_{abcd} &= {}_{abcd} + \frac{1}{4}e^{\varphi}(V_{ad}V_{bc} - V_{ac}V_{bd} - 2V_{ab}V_{cd}) \\
R_{abcy} &= \frac{1}{4}e^{\varphi}(V_{bc}\nabla_a\varphi - V_{ac}\nabla_b\varphi - 2V_{ab}\nabla_c\varphi - 2\nabla_c V_{ab}) \\
R_{aycy} &= \frac{1}{4}e^{\varphi}(e^{\varphi}V_a{}^bV_{cb} - \nabla_a\varphi\nabla_c\varphi - 2\nabla_c\nabla_a\varphi)
\end{aligned}$$

$$\begin{aligned}
\nabla_a\Phi &= \nabla_a\bar{\phi} + \frac{1}{4}\nabla_a\varphi \;;\; \nabla_y\Phi = 0 \\
H^2 &= \bar{H}_{abc}\bar{H}^{abc} + 3e^{-\varphi}W^2 \\
H_{abc} &= \bar{H}_{abc} \;;\; H_{aby} = W_{ab}
\end{aligned}$$

$$\begin{aligned}
V_{ab} &= \partial_a g_b - \partial_b g_a \\
W_{ab} &= \partial_a b_b - \partial_b b_a
\end{aligned}$$

$$\bar{H}_{abc} = \hat{H}_{abc} - \frac{١}{٢}g_aW_{bc} - \frac{١}{٢}g_cW_{ab} - \frac{١}{٢}g_bW_{ca} - \frac{١}{٢}b_aV_{bc} - \frac{١}{٢}b_cV_{ab} - \frac{١}{٢}b_bV_{ca}$$

Minimal Basis

$$\mathbf{S}_{\text{eff}} = \sum_{n=0}^{\infty} \alpha'^n \mathbf{S}_n = \mathbf{S}_0 + \alpha' \mathbf{S}_1 + \alpha'^2 \mathbf{S}_2 + \cdots \ ; \quad \mathbf{S}_n = -\frac{2}{\kappa^2} \int d^D x \sqrt{-G} e^{-2\Phi} \mathcal{L}_n$$

* In a minimal basis with independent terms (those that are not related to each other by a field redefinition or a total derivative):

All contraction of indices of all field contents	+	Field Redefinitions (using E.O.M)	+	T.D. Terms	$= \mathbf{0}$

$$\mathcal{L}_0 = a_1 R + a_2 \nabla_\alpha \Phi \nabla^\alpha \Phi + a_3 H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} \,,$$

$$\mathcal{L}_1 = b_1 R_{\alpha\beta\gamma\sigma} R^{\alpha\beta\gamma\sigma} + b_2 R_{\alpha\beta\gamma\sigma} H^{\alpha\beta\kappa} H^{\gamma\sigma}{}_{\kappa} + \cdots + b_8 (\partial_\alpha \Phi \partial^\alpha \Phi)^2 \,,$$

$$\mathcal{L}_2 = c_1 R_\alpha{}^\kappa{}_\gamma{}^\lambda R^{\alpha\beta\gamma\theta} R_{\beta\lambda\theta\kappa} + \cdots + c_{60} H_\alpha{}^{\theta\kappa} H^{\alpha\beta\gamma} \nabla_\mu H_{\gamma\kappa\lambda} \nabla^\mu H_{\beta\theta}{}^\lambda \,,$$

$$\mathcal{L}_3 = d_1 \mathcal{L}_3^{R^4} + d_2 \mathcal{L}_3^{H^8} + \cdots + d_{872} \mathcal{L}_3^{(\partial\partial\Phi)^4}$$

$$H_{abc} = \partial_a B_{bc} + \partial_c B_{ab} + \partial_b B_{ca}$$

M. R. Garousi, and H. Razaghian "*Minimal independent couplings at order α'^2*",(2019), arXiv:1905.10800.

Effective Action: Order $(\alpha')^0$

$$\mathcal{L}_0 = a_1 R + a_2 \nabla_\alpha \Phi \nabla^\alpha \Phi + a_3 H_{\alpha\beta\gamma} H^{\alpha\beta\gamma}$$

$$\boxed{S_{\text{eff}}(\psi) - S_{\text{eff}}(\psi') \;=\; \int d^{D-1}x \sqrt{-\bar{g}} \nabla_a (e^{-2\bar{\phi}} J^a)}$$

$$J^a = \sum_{n=\circ}^{\infty} \alpha'^n J^a_n \qquad \nabla_{[a} V_{bc]} = 0, \qquad \nabla_{[a} W_{bc]} = 0, \qquad \nabla_{[a} \bar{H}_{bcd]} + \frac{3}{2} V_{[ab} W_{cd]} = 0$$

$$\boxed{\mathbf{S}_0 \;=\; -\frac{2a_1}{\kappa^2} \int dx\; e^{-2\Phi} \sqrt{-G} \left(R + 4 \nabla_a \Phi \nabla^a \Phi - \frac{1}{12} H^2 \right)}$$

* For bosonic theory $a_1 = 1/4$ and for heterotic theory $a_1 = 1/8$

M. R. Garousi, "*Four-derivative couplings via T-duality constraint*", (2019), arXiv:1907.06500.

Effective Action: Higher order corrections of T-duality

Higher-order actions are ***not*** invariant under Butcher rules, and the following corrections must be added to them:

$$\boxed{\psi' = \psi'_0 + \sum_{n=1}^{\infty} \frac{\alpha'^n}{n!} \psi'_n}$$

$$\varphi' = -\varphi + \sum_{n=1}^{\infty} \frac{\alpha'^n}{n!} \Delta\varphi^{(n)} \ , \ g'_a = b_a + e^{\varphi/2} \sum_{n=1}^{\infty} \frac{\alpha'^n}{n!} \Delta g_a^{(n)} \ , \ b'_a = g_a + e^{-\varphi/2} \sum_{n=1}^{\infty} \frac{\alpha'^n}{n!} \Delta b_a^{(n)}$$

$$\bar{g}'_{ab} = \bar{g}_{ab} + \sum_{n=1}^{\infty} \frac{\alpha'^n}{n!} \Delta \bar{g}_{ab}^{(n)} \ , \ \bar{H}'_{abc} = \bar{H}_{abc} + \sum_{n=1}^{\infty} \frac{\alpha'^n}{n!} \Delta \bar{H}_{abc}^{(n)} \ , \ \bar{\phi}' = \bar{\phi} + \sum_{n=1}^{\infty} \frac{\alpha'^n}{n!} \Delta \bar{\phi}^{(n)} \ ,$$

The appearance of corrections in the action. Example:

$$S_0(\underbrace{\psi'_0 + \alpha' \psi'_1}_{\psi'}) \quad = \quad S_0(\psi'_0) + \underbrace{\alpha' \delta S_0^{(1)}}_{S^{(0,1)}(\psi'_0)} + \cdots$$

Effective Action: Higher order corrections of T-duality

Therefore, the general form of the effective action at each order n of α' can be obtained by solving the following relation:

$$\boxed{\sum_{n=0}^{\infty}\frac{\alpha'^n}{n!}S^{(n)}-\sum_{n=0}^{\infty}\frac{\alpha'^n}{n!}S^{(n)}(\psi_0')-\sum_{n=0,m=1}^{\infty}\frac{\alpha'^{n+m}}{n!m!}S^{(n,m)}(\psi_0')=\sum_{n=0}^{\infty}\frac{\alpha'^n}{n!}\int d^{25}x\sqrt{-\bar{g}}\nabla_a\left[e^{-2\bar{\phi}}J_n^a\right]}$$

Effective Action: Order α'

$$\boxed{S_1(\psi)-S_1(\psi_0')-\delta S_0^{(1)} \;=\; \int d^{D-1}x\sqrt{-\bar{g}}\nabla_a[e^{-2\bar{\phi}}J_1^a]}$$

Metsaev-Tseytlin Scheme

$$\mathbf{S}_{MT}^{(1)} \;=\; -\frac{2}{4\kappa^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\Big[\frac{1}{24}H_\alpha{}^{\delta\epsilon}H^{\alpha\beta\gamma}H_{\beta\delta}{}^{\varepsilon}H_{\gamma\epsilon\varepsilon}-\frac{1}{8}H_{\alpha\beta}{}^{\delta}H^{\alpha\beta\gamma}H_\gamma{}^{\epsilon\varepsilon}H_{\delta\epsilon\varepsilon}$$
$$+R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}-\frac{1}{2}H_\alpha{}^{\delta\epsilon}H^{\alpha\beta\gamma}R_{\beta\gamma\delta\epsilon}\Big]\,.$$

M. R. Garousi, "*Four-derivative couplings via T-duality constraint*", (2019), arXiv:1907.06500.

R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. B **293**, 385 (1987). doi:10.1016/0550-3213(87)90077-0

Effective Action: Order α'

Meissner Scheme

Starting with the maximal base (terms related only by T.D. terms):

$$\mathbf{S}_1 = \frac{-2\alpha'}{\kappa^2}\int d^D x e^{-2\Phi}\sqrt{-G}\Big[b_1 R_{abcd}R^{abcd} + \cdots + b_{20} H_f{}^{ab} H_{gab} H^{fch} H^g{}_{ch}\Big]$$

After solving the T-duality constraint for this action, 11 free parameters (up to an overall factor) will remain. With a ***specific*** choice of these coefficients, we have:

$$\begin{aligned}\mathbf{S}_1^{\mathrm{M}} = \frac{2\alpha' b_1}{\kappa^2}\int d^D x e^{-2\phi}\sqrt{-G}\Big[& -R^2_{GB} + 16\left(R^{ab} - \frac{1}{2}g^{ab}R\right)\partial_a\phi\partial_b\phi - 16\nabla^2\phi(\partial\phi)^2 \\ & + 16(\partial\phi)^4 + \frac{1}{2}\left(R_{abcd}H^{abe}H^{cd}{}_e - 2R^{ab}H^2_{ab} + \frac{1}{3}RH^2\right) - 2\left(\nabla^a\partial^b\phi H^2_{ab} - \frac{1}{3}\nabla^2\phi H^2\right) \\ & -\frac{2}{3}(\partial\phi)^2 H^2 - \frac{1}{24}H_{fgh}H^f{}_a{}^b H^g{}_b{}^c H^h{}_c{}^a + \frac{1}{8}H^2_{ab}H^{2\,ab} - \frac{1}{144}(H^2)^2\Big]\end{aligned}$$

$$R^2_{GB} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2 \qquad H^2_{ab} = H_a{}^{cd}H_{bcd}$$

M. R. Garousi, "*Four-derivative couplings via T-duality constraint*", (2019), arXiv:1907.06500.

K. A. Meissner , Phys. Lett. B **392**, 298 (1997), doi:10.1016/S0370-2693(96)01556-0, [hep-th/9610131].

Effective Action: Order $(\alpha')^2$

$$\mathcal{L}_2 = c_1 R_\alpha{}^\kappa{}_\gamma{}^\lambda R^{\alpha\beta\gamma\theta} R_{\beta\lambda\theta\kappa} + \cdots + c_{60} H_\alpha{}^{\theta\kappa} H^{\alpha\beta\gamma} \nabla_\mu H_{\gamma\kappa\lambda} \nabla^\mu H_{\beta\theta}{}^\lambda \,,$$

$$\boxed{S_2(\psi) - S_2(\psi_0') - \delta S_0^{(2)} - \delta S_1^{(1)} \;=\; \int d^{D-1}x\sqrt{-\bar{g}}\nabla_a[e^{-2\bar{\phi}} J_2^a]}$$

$$\begin{aligned} S_0(\psi_0' + \alpha'\psi_1' + \alpha'^2\psi_2') &= S_0(\psi_0') + \alpha'\delta S_0^{(1)} + \alpha'^2\delta S_0^{(2)} + \cdots \\ \alpha' S_1(\psi_0' + \alpha'\psi_1') &= \alpha' S_1(\psi_0') + \alpha'^2\delta S_1^{(1)} + \cdots \end{aligned}$$

$$\mathbf{S}_M^{(2)B} = \frac{2\alpha'^2 c_1^2}{\kappa^2}\int d^{26}x\sqrt{-G}e^{-2\Phi}\Big[\frac{1}{12}H_\alpha{}^{\delta\epsilon}H^{\alpha\beta\gamma}H_{\beta\delta}{}^\zeta H_\gamma{}^{\iota\kappa}H_{\epsilon\iota}{}^\mu H_{\zeta\kappa\mu} + \cdots$$

* The fixed action has **27 terms** for both M-T and M schemes.

M. R. Garousi, "*Effective action of bosonic string theory at order α'^2*", (2019), arXiv:1907.06500.

Effective Action: **Order $(\alpha')^2$**

* The basis that led to these 27 terms is ***not unique***.

* One can choose 60 terms on **another** basis in such a way that, after fixing the coefficients, fewer terms remain:

$$\boxed{\text{178 terms}} + \boxed{\text{F.D.}} + \boxed{\text{T.D.}} + \boxed{\text{Bianchi}} = \boxed{\text{27 terms}}$$

* **Trivial solution:** all free coefficients are set to **zero**.

* **Specific** choice for coefficients, reduces the number of independent terms from 27 to:

17 for Metsaev-Tseytlin Scheme, and 12 for Meissner Scheme

H. Gholian, M. R. Garousi "*More on closed string effective actions at order α'^2*", (2023), arXiv:2311.05207.

Metsaev-Tseytlin Scheme

$$
\begin{aligned}
\mathbf{S}^{(2)}_{MT} = & -\frac{2}{16\kappa^2}\int d^{26}x\sqrt{-G}e^{-2\Phi}\Big[-\frac{4}{3}R_{\alpha}{}^{\epsilon}{}_{\gamma}{}^{\varepsilon}R^{\alpha\beta\gamma\delta}R_{\beta\varepsilon\delta\epsilon}+\frac{4}{3}R_{\alpha\beta}{}^{\epsilon\varepsilon}R^{\alpha\beta\gamma\delta}R_{\gamma\epsilon\delta\varepsilon} \\
& -\frac{1}{12}H_{\alpha}{}^{\delta\epsilon}H^{\alpha\beta\gamma}H_{\beta\delta}{}^{\varepsilon}H_{\gamma}{}^{\mu\theta}H_{\epsilon\mu}{}^{\lambda}H_{\varepsilon\theta\lambda}+\frac{1}{4}H_{\alpha\beta}{}^{\delta}H^{\alpha\beta\gamma}H_{\gamma}{}^{\epsilon\varepsilon}H_{\delta}{}^{\mu\theta}H_{\epsilon\mu}{}^{\lambda}H_{\varepsilon\theta\lambda} \\
& -\frac{1}{6}H_{\alpha\beta}{}^{\delta}H^{\alpha\beta\gamma}H_{\gamma}{}^{\epsilon\varepsilon}H_{\delta}{}^{\mu\theta}H_{\epsilon\varepsilon}{}^{\lambda}H_{\mu\theta\lambda}-2H_{\alpha}{}^{\delta\epsilon}H^{\alpha\beta\gamma}R_{\beta\delta}{}^{\varepsilon\mu}R_{\gamma\varepsilon\epsilon\mu}+\frac{1}{4}H_{\alpha\beta}{}^{\delta}H^{\alpha\beta\gamma}H_{\epsilon\varepsilon}{}^{\theta}H^{\epsilon\varepsilon\mu}R_{\gamma\mu\delta\theta} \\
& +2H^{\alpha\beta\gamma}H^{\delta\epsilon\varepsilon}R_{\alpha\beta\delta}{}^{\mu}R_{\gamma\mu\epsilon\varepsilon}-2H_{\alpha}{}^{\delta\epsilon}H^{\alpha\beta\gamma}R_{\beta}{}^{\varepsilon}{}_{\gamma}{}^{\mu}R_{\delta\varepsilon\epsilon\mu}+3H_{\alpha\beta}{}^{\delta}H^{\alpha\beta\gamma}R_{\gamma}{}^{\epsilon\varepsilon\mu}R_{\delta\varepsilon\epsilon\mu} \\
& +4H_{\alpha\beta}{}^{\delta}H^{\alpha\beta\gamma}H_{\gamma}{}^{\epsilon\varepsilon}H_{\epsilon}{}^{\mu\theta}R_{\delta\mu\varepsilon\theta}-2H_{\alpha\beta}{}^{\delta}H^{\alpha\beta\gamma}H_{\gamma}{}^{\epsilon\varepsilon}H_{\delta}{}^{\mu\theta}R_{\epsilon\mu\varepsilon\theta}-H_{\alpha}{}^{\delta\epsilon}H^{\alpha\beta\gamma}\nabla_{\delta}H_{\beta}{}^{\varepsilon\mu}\nabla_{\epsilon}H_{\gamma\varepsilon\mu} \\
& +\frac{1}{36}H^{\alpha\beta\gamma}H^{\delta\epsilon\varepsilon}\nabla_{\mu}H_{\delta\epsilon\varepsilon}\nabla^{\mu}H_{\alpha\beta\gamma}-H^{\alpha\beta\gamma}H^{\delta\epsilon\varepsilon}\nabla_{\varepsilon}H_{\gamma\epsilon\mu}\nabla^{\mu}H_{\alpha\beta\delta}-\frac{3}{4}H_{\alpha}{}^{\delta\epsilon}H^{\alpha\beta\gamma}\nabla_{\mu}H_{\delta\epsilon\varepsilon}\nabla^{\mu}H_{\beta\gamma}{}^{\varepsilon} \\
& +\frac{1}{2}H_{\alpha\beta}{}^{\delta}H^{\alpha\beta\gamma}\nabla_{\mu}H_{\delta\epsilon\varepsilon}\nabla^{\mu}H_{\gamma}{}^{\epsilon\varepsilon}\Big].
\end{aligned}
$$

Meissner Scheme

$$
\begin{aligned}
\mathbf{S}^{(2)}_{M} = & -\frac{2}{16\kappa^2}\int d^{26}x\sqrt{-G}e^{-2\Phi}\Big[-\frac{4}{3}R_{\alpha}{}^{\kappa}{}_{\gamma}{}^{\lambda}R^{\alpha\beta\gamma\theta}R_{\beta\lambda\theta\kappa}+\frac{4}{3}R_{\alpha\beta}{}^{\kappa\lambda}R^{\alpha\beta\gamma\theta}R_{\gamma\kappa\theta\lambda} \\
& -\frac{1}{12}H_{\alpha}{}^{\theta\kappa}H^{\alpha\beta\gamma}H_{\beta\theta}{}^{\lambda}H_{\gamma}{}^{\mu\nu}H_{\kappa\mu}{}^{\tau}H_{\lambda\nu\tau}+\frac{1}{4}H_{\alpha\beta}{}^{\theta}H^{\alpha\beta\gamma}H_{\gamma}{}^{\kappa\lambda}H_{\theta}{}^{\mu\nu}H_{\kappa\mu}{}^{\tau}H_{\lambda\nu\tau} \\
& +\frac{1}{48}H_{\alpha\beta}{}^{\theta}H^{\alpha\beta\gamma}H_{\gamma}{}^{\kappa\lambda}H_{\theta}{}^{\mu\nu}H_{\kappa\lambda}{}^{\tau}H_{\mu\nu\tau}-2H_{\alpha}{}^{\theta\kappa}H^{\alpha\beta\gamma}R_{\beta\theta}{}^{\lambda\mu}R_{\gamma\lambda\kappa\mu}-H_{\alpha\beta}{}^{\theta}H^{\alpha\beta\gamma}R_{\gamma}{}^{\kappa\lambda\mu}R_{\theta\lambda\kappa\mu} \\
& +2H^{\alpha\beta\gamma}H^{\theta\kappa\lambda}R_{\alpha\beta\theta}{}^{\mu}R_{\gamma\mu\kappa\lambda}-2H_{\alpha}{}^{\theta\kappa}H^{\alpha\beta\gamma}R_{\beta}{}^{\lambda}{}_{\gamma}{}^{\mu}R_{\theta\lambda\kappa\mu}+\frac{1}{4}H^{\alpha\beta\gamma}H^{\theta\kappa\lambda}\nabla_{\gamma}H_{\kappa\lambda\mu}\nabla_{\theta}H_{\alpha\beta}{}^{\mu} \\
& +\frac{1}{2}H_{\alpha}{}^{\theta\kappa}H^{\alpha\beta\gamma}\nabla_{\kappa}H_{\theta\lambda\mu}\nabla^{\mu}H_{\beta\gamma}{}^{\lambda}+H_{\alpha}{}^{\theta\kappa}H^{\alpha\beta\gamma}\nabla_{\mu}H_{\gamma\kappa\lambda}\nabla^{\mu}H_{\beta\theta}{}^{\lambda}\Big].
\end{aligned}
$$

H. Gholian, M. R. Garousi "*More on closed string effective actions at order* α'^2", (2023), arXiv:2311.05207.

Effective Action: **Order $(\alpha')^3$**

$$\boxed{S_3(\psi) - S_3(\psi'_0) - \delta S_0^{(3)} - \delta S_1^{(2)} - \delta S_2^{(1)} = \sum_{n=0}^{\infty} \int d^{25}x\sqrt{-\bar{g}}\; \nabla_a \left[e^{-2\bar{\phi}} J_n^a\right]}$$

* After solving the T-duality constraint, one gets:

$$\begin{aligned}
\mathbf{S}_{MT}^{(3)} \;=\; & -\frac{2}{\kappa^2}\int d^{26}x\sqrt{-G}e^{-2\Phi}\Big[\frac{1}{16}R_{\alpha\beta}{}^{\epsilon\varepsilon}R^{\alpha\beta\gamma\delta}R_{\gamma}{}^{\zeta}{}_{\epsilon}{}^{\mu}R_{\delta\zeta\varepsilon\mu} + (\frac{1}{72} - \frac{a}{9})R_{\alpha}{}^{\epsilon}{}_{\gamma}{}^{\varepsilon}R^{\alpha\beta\gamma\delta}R_{\beta}{}^{\zeta}{}_{\epsilon}{}^{\mu}R_{\delta\mu\varepsilon\zeta} \\
& + \frac{1}{18}(-1-a)R_{\alpha\beta}{}^{\epsilon\varepsilon}R^{\alpha\beta\gamma\delta}R_{\gamma}{}^{\zeta}{}_{\epsilon}{}^{\mu}R_{\delta\mu\varepsilon\zeta} - \frac{1}{4}R_{\alpha\gamma\beta}{}^{\epsilon}R^{\alpha\beta\gamma\delta}R_{\delta}{}^{\varepsilon\zeta\mu}R_{\epsilon\zeta\varepsilon\mu} \\
& - \frac{1}{4}H_{\alpha\epsilon}{}^{\mu}R^{\alpha\beta\gamma\delta}R_{\gamma}{}^{\epsilon\varepsilon\zeta}\nabla_{\beta}\nabla_{\zeta}H_{\delta\varepsilon\mu} + \cdots\Big],
\end{aligned}$$

$$\begin{aligned}
\mathbf{S}_{M}^{(3)} \;=\; & -\frac{2}{\kappa^2}\int d^{26}x\sqrt{-G}e^{-2\Phi}\Big[\frac{1}{16}R_{\alpha\beta}{}^{\epsilon\varepsilon}R^{\alpha\beta\gamma\delta}R_{\gamma}{}^{\zeta}{}_{\epsilon}{}^{\mu}R_{\delta\zeta\varepsilon\mu} + (\frac{1}{72} - \frac{a}{9})R_{\alpha}{}^{\epsilon}{}_{\gamma}{}^{\varepsilon}R^{\alpha\beta\gamma\delta}R_{\beta}{}^{\zeta}{}_{\epsilon}{}^{\mu}R_{\delta\mu\varepsilon\zeta} \\
& + \frac{1}{18}(-1-a)R_{\alpha\beta}{}^{\epsilon\varepsilon}R^{\alpha\beta\gamma\delta}R_{\gamma}{}^{\zeta}{}_{\epsilon}{}^{\mu}R_{\delta\mu\varepsilon\zeta} + \frac{3}{4}R_{\alpha\gamma\beta}{}^{\epsilon}R^{\alpha\beta\gamma\delta}R_{\delta}{}^{\varepsilon\zeta\mu}R_{\epsilon\zeta\varepsilon\mu} \\
& - \frac{1}{16}R_{\alpha\gamma\beta\delta}R^{\alpha\beta\gamma\delta}R_{\epsilon\zeta\varepsilon\mu}R^{\epsilon\varepsilon\zeta\mu} + \cdots\Big],
\end{aligned}$$

* The parameter $\boldsymbol{a}$ remains as the sole unfixed parameter among the <u>872</u> parameters.

Effective Action: **Order $(\alpha')^3$**

* The effective action at the 8th derivative order has the general form of:

$$\mathcal{L}_3 = d_1 \left(R^{\alpha\beta\mu\nu} R_{\mu\nu}{}^{\gamma\delta} R_{\alpha\gamma}{}^{\rho\sigma} R_{\rho\sigma\beta\delta} + \ldots\right) + d_2 \left(R^{\alpha\beta\mu\nu} R_{\mu\nu}{}^{\gamma\delta} R_{\alpha\gamma}{}^{\rho\sigma} R_{\rho\sigma\beta\delta} + \ldots\right)$$

* Some of the terms contribute to the lower orders, while others specifically correspond to the $\mathcal{N} = 2$ **superstring theory.**

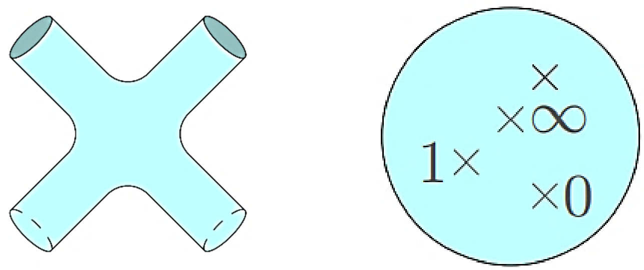
How to fix the parameter $\boldsymbol{a}$?

Superstring theory: Lacks effective actions at orders α' and α'^2 , only terms with coefficient $\boldsymbol{a}$ survive. Comparison with the **4-point S-matrix** element fixes this parameter to be proportional to $\zeta(3)$.

Bosonic string theory: One cannot simply assume that $\boldsymbol{a}$ must similarly be proportional to $\zeta(3)$. Instead, the parameter $\boldsymbol{a}$ must be determined by explicitly comparing the *gravitational couplings* with the **4-point** S-matrix element in the bosonic string theory.

S-Matrix Constraint

Kawai-Lewellen-Tye (KLT) Method: the sphere-level closed string S-matrix element can be expressed in terms of disk-level S-matrix elements of open strings:



* The amplitude for four **gauge boson** vertex operators is (*Veneziano* amplitude):

$$\mathcal{A}(\alpha' s, \alpha' t) \;\sim\; \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1+\alpha' u)} K$$

$$s = -(k_1 + k_2)^2, \quad t = -(k_1 + k_4)^2, \quad u = -(k_1 + k_3)^2; \quad s + t + u = 0$$

$$K^B(\alpha') = \left(\frac{\alpha'^2 st}{1+\alpha' u}\right)\zeta_1\cdot\zeta_3\,\zeta_2\cdot\zeta_4 + \left(\frac{\alpha'^2 su}{1+\alpha' t}\right)\zeta_2\cdot\zeta_3\,\zeta_1\cdot\zeta_4 + \left(\frac{\alpha'^2 tu}{1+\alpha' s}\right)\zeta_1\cdot\zeta_2\,\zeta_3\cdot\zeta_4 + \cdots$$

* These kinematic factors are ***stu-symmetric.***

S-Matrix Constraint

$$A_{\text{closed}} = A_{\text{open}} \times \bar{A}_{\text{open}}$$

$$\begin{aligned} A &= -\left(\frac{\kappa^2}{\pi\alpha'}\right)\sin\left(\frac{\alpha'\pi}{2}k_2\cdot k_3\right)\mathcal{A}\left(\frac{\alpha'}{4}s,\frac{\alpha'}{4}t\right)\bar{\mathcal{A}}\left(\frac{\alpha'}{4}t,\frac{\alpha'}{4}u\right), \\ &= -\left(\frac{\kappa^2}{\alpha'}\right)\frac{\Gamma(-\frac{\alpha'}{4}s)\Gamma(-\frac{\alpha'}{4}t)\Gamma(-\frac{\alpha'}{4}u)}{\Gamma(1+\frac{\alpha'}{4}s)\Gamma(1+\frac{\alpha'}{4}t)\Gamma(1+\frac{\alpha'}{4}u)}K^B\left(\frac{\alpha'}{4}\right)\bar{K}^B\left(\frac{\alpha'}{4}\right) \end{aligned}$$

The Gamma functions have massless poles and an infinite tower of massive poles:

$$\frac{\Gamma(-\frac{\alpha'}{4}s)\Gamma(-\frac{\alpha'}{4}t)\Gamma(-\frac{\alpha'}{4}u)}{\Gamma(1+\frac{\alpha'}{4}s)\Gamma(1+\frac{\alpha'}{4}t)\Gamma(1+\frac{\alpha'}{4}u)} = -\frac{64}{\alpha'^3 stu} - 2\zeta(3) + \cdots$$

higher-order terms in α'

For simplicity, we restrict our attention to **single-trace** terms of the form $\mathbf{Tr}(\epsilon\epsilon\epsilon\epsilon)$

S-Matrix Constraint

From KLT, finally one gets:

$$A = f(s,t,u)\mathrm{Tr}(\epsilon_1\epsilon_3\epsilon_2\epsilon_4) + f(u,t,s)\mathrm{Tr}(\epsilon_1\epsilon_2\epsilon_3\epsilon_4) + f(t,u,s)\mathrm{Tr}(\epsilon_1\epsilon_2\epsilon_4\epsilon_3)$$

with $\epsilon_i = \zeta_i\overline{\zeta_i}$ and the functions $f(s,t,u)$ given by

$$f(s,t,u) = \frac{\kappa^2}{2}\Big[s + \frac{\alpha'}{4}s^2 + \frac{\alpha'^2}{16}(s^3 - stu) + \frac{\alpha'^3}{64}s^2\Big(s^2 + 2st + 2t^2 + \zeta(3)tu\Big) + \cdots\Big]$$

Now we compute the four-graviton S-matrix element in low-energy **field theory** up to order α'^3 (in the Meissner scheme). The metric perturbation is given by:

$$G_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

The four-graviton Feynman amplitude consists of both **contact terms** and **massless pole** contributions, where the latter involve graviton and dilaton propagation between vertices.

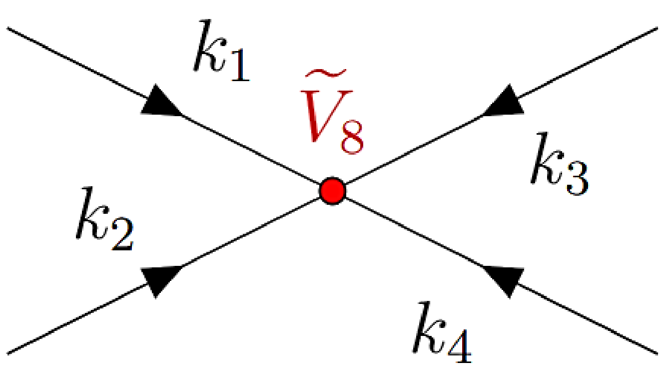
S-Matrix Constraint

All vertices and propagators must be computed in the Einstein frame, where the Einstein-Hilbert term has no overall dilaton factor. Notably, for the single-trace terms, only the **graviton** propagates between vertices:

$$(\tilde{G}_h)_{\mu\nu,\lambda\rho} \;=\; \frac{1}{2k^2}\left(\eta_{\mu\lambda}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\lambda}-\frac{1}{\frac{D}{2}-1}\eta_{\mu\nu}\eta_{\lambda\rho}\right)$$

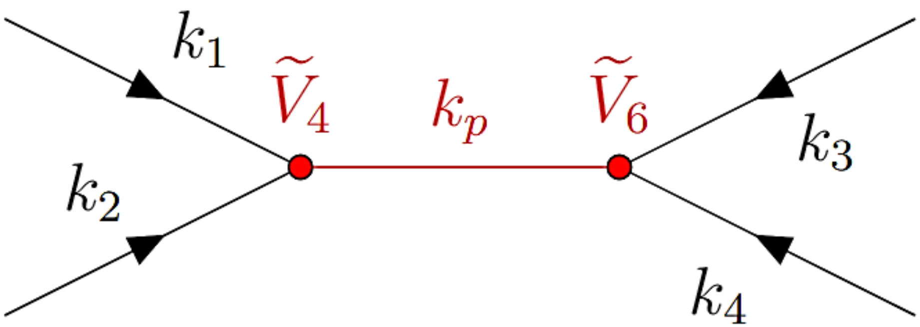
Using the gravity couplings at order α'^3, we get the following contact term:

$$\begin{aligned}A_{\rm contact} \;=\; \frac{\alpha'^3}{576}\Big[&(-18s^4-2(31+4a)s^3t-(89+8a)s^2t^2-54st^3-27t^4)\mathrm{Tr}(\epsilon_1\epsilon_3\epsilon_2\epsilon_4)\\ &+(-18s^4+2(-5+4a)s^3t+(-11+16a)s^2t^2+2(-5+4a)st^3-18t^4)\mathrm{Tr}(\epsilon_1\epsilon_2\epsilon_3\epsilon_4)\\ &+(-27s^4-54s^3t-(89+8a)s^2t^2-2(31+4a)st^3-18t^4)\mathrm{Tr}(\epsilon_1\epsilon_2\epsilon_4\epsilon_3)\Big]\,.\end{aligned}$$



S-Matrix Constraint

$$A_{\text{pole}} = \widetilde{V}(hhh)\widetilde{G}_h\widetilde{V}(hhh) = \frac{3\alpha'^2}{128}\Big[\left(s^4+4s^3t+6s^2t^2+4st^3+2t^4\right)\mathrm{Tr}(\epsilon_1\epsilon_3\epsilon_2\epsilon_4)$$
$$+\left(s^4+t^4\right)\mathrm{Tr}(\epsilon_1\epsilon_2\epsilon_3\epsilon_4)$$
$$+\left(2s^4+4s^3t+6s^2t^2+4st^3+t^4\right)\mathrm{Tr}(\epsilon_1\epsilon_2\epsilon_4\epsilon_3)\Big]\,.$$



- The combination $\boldsymbol{A}_{\text{contact}} + \boldsymbol{A}_{\text{Pole}}$ must reproduce the KLT string amplitude.
- This condition fixes the parameter a to the value:

$$\boxed{a = \frac{1}{8} - \frac{9}{8}\zeta(3)\,.}$$

Final Result

Metsaev-Tseytlin Scheme

$$\begin{aligned}\mathbf{S}^{(3)}_{MT} = & -\frac{2}{\kappa^2}\int d^{26}x\sqrt{-G}e^{-2\Phi}\Big[\frac{\zeta(3)}{16}\Big(R_{\alpha\beta}{}^{\epsilon\varepsilon}R^{\alpha\beta\gamma\delta}R_{\gamma}{}^{\zeta}{}_{\epsilon}{}^{\mu}R_{\delta\mu\varepsilon\zeta}+2R_{\alpha}{}^{\epsilon}{}_{\gamma}{}^{\varepsilon}R^{\alpha\beta\gamma\delta}R_{\beta}{}^{\zeta}{}_{\epsilon}{}^{\mu}R_{\delta\mu\varepsilon\zeta}\Big)\\ & +\frac{1}{16}R_{\alpha\beta}{}^{\epsilon\varepsilon}R^{\alpha\beta\gamma\delta}R_{\gamma}{}^{\zeta}{}_{\epsilon}{}^{\mu}R_{\delta\zeta\varepsilon\mu}-\frac{1}{4}R_{\alpha\gamma\beta}{}^{\epsilon}R^{\alpha\beta\gamma\delta}R_{\delta}{}^{\varepsilon\zeta\mu}R_{\epsilon\zeta\varepsilon\mu}-\frac{1}{16}R_{\alpha\beta}{}^{\epsilon\varepsilon}R^{\alpha\beta\gamma\delta}R_{\gamma}{}^{\zeta}{}_{\epsilon}{}^{\mu}R_{\delta\mu\varepsilon\zeta}\\ & -\frac{1}{4}H_{\alpha\epsilon}{}^{\mu}R^{\alpha\beta\gamma\delta}R_{\gamma}{}^{\epsilon\varepsilon\zeta}\nabla_{\beta}\nabla_{\zeta}H_{\delta\varepsilon\mu}+\cdots\Big],\end{aligned}$$

Meissner Scheme

$$\begin{aligned}\mathbf{S}^{(3)}_{M} = & -\frac{2}{\kappa^2}\int d^{26}x\sqrt{-G}e^{-2\Phi}\Big[\frac{1}{16}R_{\alpha\beta}{}^{\epsilon\varepsilon}R^{\alpha\beta\gamma\delta}R_{\gamma}{}^{\zeta}{}_{\epsilon}{}^{\mu}R_{\delta\zeta\varepsilon\mu}+\frac{3}{4}R_{\alpha\gamma\beta}{}^{\epsilon}R^{\alpha\beta\gamma\delta}R_{\delta}{}^{\varepsilon\zeta\mu}R_{\epsilon\zeta\varepsilon\mu}\\ & -\frac{1}{16}R_{\alpha\gamma\beta\delta}R^{\alpha\beta\gamma\delta}R_{\epsilon\zeta\varepsilon\mu}R^{\epsilon\varepsilon\zeta\mu}-\frac{1}{16}R_{\alpha\beta}{}^{\epsilon\varepsilon}R^{\alpha\beta\gamma\delta}R_{\gamma}{}^{\zeta}{}_{\epsilon}{}^{\mu}R_{\delta\mu\varepsilon\zeta}\\ & +\frac{\zeta(3)}{16}\Big(R_{\alpha\beta}{}^{\epsilon\varepsilon}R^{\alpha\beta\gamma\delta}R_{\gamma}{}^{\zeta}{}_{\epsilon}{}^{\mu}R_{\delta\mu\varepsilon\zeta}+2R_{\alpha}{}^{\epsilon}{}_{\gamma}{}^{\varepsilon}R^{\alpha\beta\gamma\delta}R_{\beta}{}^{\zeta}{}_{\epsilon}{}^{\mu}R_{\delta\mu\varepsilon\zeta}\Big)+\cdots\Big].\end{aligned}$$

Thank You