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JT Gravity and the Schwarzian Path Integral

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Outline

Why 2 dimensions?

Riemann tensor has $\frac{D^2(D^2-1)}{12}$ components in D dimensions:

- 11D: 1210 (1144 Weyl and 66 Ricci)
- 10D: 825 (770 Weyl and 55 Ricci)
- 5D: 50 (35 Weyl and 15 Ricci)
- 4D: 20 (10 Weyl and 10 Ricci)
- 3D: 6 (Ricci)
- 2D: 1 (Ricci scalar)
- 2D: lowest dimension exhibiting black holes (BHs)
- Simplest gravitational theories with BHs in 2D
- No Einstein gravity in 2D

2d gravity model

- in two spacetime dimensions, the simplest candidate model would be Einstein–Hilbert gravity:

$$I = \underbrace{-\frac{S_0}{2\pi}\left[\frac{1}{2}\int_{\mathcal{M}}\sqrt{g}R + \int_{\partial\mathcal{M}}\sqrt{h}K\right]}_{\text{topological term } = S_0\,\chi(\mathcal{M})} \tag{1}$$

- This model is topological since its Euclidean action is just the Euler characteristic, and the Einstein tensor vanishes identically
- To get something more interesting, introduce a direct coupling of the Ricci scalar to a new scalar field ϕ_0

JT gravity

- The most general action for the bulk is:

$$I_b = -\frac{1}{2}\int d^2x\sqrt{g}\left(U(\phi_0)R + V(\phi_0)g^{\alpha\beta}\partial_\alpha\phi_0\partial_\beta\phi_0 + W(\phi_0)\right) \quad (2)$$

- It is possible to eliminate two of the three functions by reparametrizing ϕ_0 and field redefinition so the action reduce to: (Witten,2020)

$$I_b = -\frac{1}{2}{}^2 x\sqrt{g}\,(\phi R + W(\phi)) \quad (3)$$

- JT gravity is a model which $W(\phi) = 2\phi$

- The action of JT gravity is:

$$I = \mathrm{S}_0\, \chi(\mathcal{M}) - \Bigg[\underbrace{\frac{1}{2}\int_{\mathcal{M}} \sqrt{g}\phi(R+2)}_{\text{sets } R=-2} + \underbrace{\int_{\partial\mathcal{M}} \sqrt{h}\phi(K-1)}_{\text{gives action for boundary}} \Bigg] \tag{4}$$

- According to holography, quantum gravity in AdS_2 should be dual to a quantum mechanical theory living on the boundary
- The boundary of the hyperbolic disk is a circle and the path integral of a quantum system on a circle is a thermal partition function (Maldacena,2003)
- One can doing the JT path integral in two steps:
  1- Integrate over the ϕ which setting $R = -2$ everywhere
  2- Integrate over the "cutout shape" of the geometry within the hyperbolic disk(at genus zero)
- The resulting path integral is the partition function of the Schwarzian theory

Schwarzian theory – Partition function

- Partition function for the topology of the disk is:

$$Z(\beta) = e^{S_0} \int [dg\, d\Phi]\ e^{\frac{1}{2}\int_M d^2x\sqrt{g}\Phi(R+2)+\oint_{\partial M}\sqrt{h}\Phi(K-1)} \tag{5}$$

$$= e^{S_0} \int [df]\ e^{\Phi_r \int_0^\beta d\tau \{\tan\frac{\pi f(\tau)}{\beta},\tau\}} \tag{6}$$

- The measure can be derived from the BF analysis

$$\Omega = 2\int \mathrm{Tr}\left[\delta_1 A \wedge \delta_2 A\right] \tag{7}$$

- From this symplectic form one can derive a measure over the Schwarzian mode by translating the boundary curve into first-order formalism

The basic objects

- The one forms:

$$e^a = e^a_\mu dx^\mu$$

- The spin connection:

$$\omega^a{}_b = \epsilon^a{}_b \omega$$

- The no-torsion condition:

$$de^a + \omega^a{}_b \wedge e^b = 0$$

- The curvature two form:

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = d\omega^a{}_b$$

- The volume element:

$$e^1 \wedge e^2 = \sqrt{g} d^2 x \qquad \sqrt{g} d^2 x R = 2 d\omega^1{}_2 = 2 d\omega$$

JT action in the first order formalism

- We can write the JT action in the first order formalism:

$$\frac{1}{2}\int \sqrt{g}\phi(R+2) \to \int \left[\phi(d\omega + e^1 \wedge e^2) + \phi_a(de^a + \epsilon^a{}_b \omega \wedge e^b)\right] \quad (8)$$

- We can write the action in terms of a matrix of scalars B and a matrix of one forms A:

$$I_{JT} = i\int \mathrm{Tr}\,(BF) \quad (9)$$

$$B = -i\begin{pmatrix} -\phi^1 & \phi^2 + \phi \\ \phi^2 - \phi & \phi^1 \end{pmatrix} \qquad A = \frac{1}{2}\begin{pmatrix} -e^1 & e^2 - \omega \\ e^2 + \omega & e^1 \end{pmatrix}$$

- The action can now be recognized as an $SL(2,\mathbb{R})$ "BF" theory

Symplectic form

- The path integral on orientable surfaces for BF theories with compact gauge group reduces to an integral over flat connections with the measure induced by the symplectic form on the space of gauge fields (Witten,1992)

$$\Omega(\sigma,\eta) = 2\alpha \int \mathrm{Tr}\left(\sigma \wedge \eta\right). \tag{10}$$

- σ and η are elements of the tangent space in the space of gauge fields
- They are one forms that parametrize infinitesimal variations of A
- The constant α is arbitrary and can be absorbed into a shift of S_0

- A symplectic form is a two-form, which is supposed to take as input two vectors in the tangent space to this point
- The tangent space can be described as consisting of gauge fields δA

$$d(\delta A) + A \wedge \delta A + \delta A \wedge A = 0 \tag{11}$$

- We can stick two such configurations into (10) and get a number $\Omega(\delta_1 A, \delta_2 A)$, so (10) is a two form in that sense
- For a gauge transformation of one of the deformations: $\delta_2 A \to \delta_2 A + d\Theta + [A, \Theta]$ the value of $\Omega(\delta_1 A, \delta_2 A)$ change in the value of

$$2\alpha \int \mathrm{Tr}\left(\delta_1 A \wedge (d\Theta + [A, \Theta])\right) \tag{12}$$

- So $\Omega(\delta_1 A, \delta_2 A)$ is gauge-invariant on the space of flat connections

Boundary condition

- The boundary condition in second-order formulation:

$$g_{uu}\big|_{\mathsf{bdy}} = \frac{1}{\epsilon^2}, \qquad \phi\big|_{\mathsf{bdy}} = \frac{\gamma}{\epsilon}, \qquad \epsilon \to 0 \tag{13}$$

- The u coordinate is the time in boundary theory and it should run from zero to β
- The total length of the boundary is β/ϵ
- This type of boundary condition allows a "boundary graviton" mode corresponding to a wiggly boundary (MaldacenaStanford,Yang,216)

- Euclidean AdS_2 in global coordinates:

$$ds^2 = d\rho^2 + \sinh^2(\rho) d\theta^2 \tag{14}$$

- $\theta(u)$ specifies the angle in the hyperbolic plane

- The function $\rho(u)$ is fixed by (13)

- Asymptotic behavior of the metric$\rightarrow$ B.C in the BF theory

- We can use coordinates r, u where r is a coordinate that measures distance to the boundary (which is taken to $r = \infty$ in the $\epsilon \rightarrow 0$ limit)

- One can show that a wiggly boundary leads to a metric with the large r behavior:(SSS,2019)

$$ds^2 = dr^2 + \left(\tfrac{1}{4}e^{2r} - \mathsf{Sch}(u) + \ldots\right) du^2 \tag{15}$$

- The extrinsic curvature of a constant r surface for large r is:

$$K = \frac{1}{2}g^{uu}\partial_r g_{uu} = \frac{\frac{1}{2}\partial_r\left(\frac{1}{4}e^{2r} - \mathsf{Sch}(u) + \ldots\right)}{\frac{1}{4}e^{2r} - \mathsf{Sch}(u) + \ldots} = 1 + 4e^{-2r}\mathsf{Sch}(u) + \ldots \tag{16}$$

- The action with an appropriate boundary term:

$$I = -i \int_{\mathcal{M}} \mathrm{Tr}(BF) + \frac{i}{2} \int_{\partial \mathcal{M}} \mathrm{Tr}\,(BA) \tag{17}$$

- Standard minimal boundary condition for BF theory would be to fix some linear combination of B and A_u to zero:

$$B + icA_u\big|_{\mathsf{bdy}} = 0, \qquad c = \text{const} \tag{18}$$

- Asymptotic conditions (15) can be written in first-order variables as:

$$e^1 = dr,\ e^2 = \left(\tfrac{1}{2}e^r - \mathsf{Sch}(u)e^{-r}\right) du,\ \omega = -\left(\tfrac{1}{2}e^r + \mathsf{Sch}(u)e^{-r}\right) du \tag{19}$$

- In terms of the gauge field:

$$A = \frac{dr}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{du}{2} \begin{pmatrix} 0 & e^r \\ -2\,\mathsf{Sch}(u)e^{-r} & 0 \end{pmatrix} \tag{20}$$

- Sch(u) is an arbitrary function which describes the freedom in A
- We should check that the action for this quantity agrees with expectations from JT gravity
- $\phi = \frac{\gamma}{\epsilon} = \gamma \frac{e^r}{2}$
- This fixes $c = 2\gamma$, so the boundary term becomes:
- $$I = \gamma \int du \, \mathrm{Tr}\left(A_u^2\right) = -\gamma \int du \, \mathsf{Sch}(u) \tag{21}$$
- It is the same Schwarzian action one gets from the standard treatment of JT

- In BF theory the physical mode is a transformation by a gauge transformation that does not vanish at infinity, but instead limits to a function $\Theta(u,r)$
- In principle $\Theta(u,r)$ is an $sl(2,\mathbb{R})$ element
- The most general transformation for some function $\varepsilon(u)$ has the large r behavior:

$$\Theta(u,r) \to \begin{pmatrix} \frac{1}{2}\varepsilon'(u) & \frac{1}{2}e^{r}\varepsilon(u) \\ -e^{-r}\left[\text{Sch}(u)\varepsilon(u)+\varepsilon''(u)\right] & -\frac{1}{2}\varepsilon'(u) \end{pmatrix} \tag{22}$$

- This preserves (20) but induces the transformation:

$$\text{Sch}(u) \to \text{Sch}(u) + \varepsilon'''(u) + \varepsilon(u)\text{Sch}'(u) + 2\varepsilon'(u)\text{Sch}(u) \tag{23}$$

- This is the transformation of the Schwarzian derivative $\text{Sch}(f(u),u)$ under an infinitesimal reparametrization

Measure

- One can evaluate the measure for the wiggles, by evaluating the symplectic form

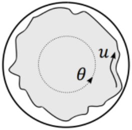
$$\Omega(\delta_1 A, \delta_2 A) = 2\alpha \int_{\mathcal{M}} \mathrm{Tr}(\delta_1 A \wedge \delta_2 A) \tag{24}$$

on a pair of configurations where $\delta_i A = d\Theta_i + [A, \Theta_i]$

- For any formal gauge transformations Θ_i, the integrand in (24) is :(SSS,2019)

$$\Omega = \frac{\alpha}{2} \int_0^\beta du \Big[d\varepsilon'(u) \wedge d\varepsilon''(u) - 2\,\mathrm{Sch}(u)\, d\varepsilon(u) \wedge d\varepsilon'(u) \Big] \tag{25}$$

- Boundary on the disk is corresponding to a wiggly boundary



(26)

- The path integrals is one-loop exact (Stanford,Witten,2017)
- So we can evaluate exactly by doing the path integral for small fluctuations
- The JT action reduces to the boundary extrinsic curvature term in which is:

$$-\gamma \int du\, \mathrm{Sch}(\tan\frac{\theta}{2}, u) \tag{27}$$

- The Schwarzian path integral is:

$$Z_{\mathsf{Sch}}^{\mathsf{disk}}(\beta) = \int \frac{d\mu[\theta]}{SL(2,\mathbb{R})} \exp\left[-\frac{\gamma}{2}\int_0^\beta du \left(\frac{\theta''^2}{\theta'^2} - \theta'^2\right)\right] \tag{28}$$

- The measure $d\mu[\theta]$ means the measure induced by the symplectic form (25)
- Using the one-loop exactness of this integral, we can get the exact answer by doing the Gaussian integral for small fluctuations about the classical solution

$$\theta(u) = \frac{2\pi}{\beta}\left(u + \varepsilon(u)\right) \tag{29}$$

Recipe for calculation

- Recipe for calculation based on: (SSS,2019)

- Expand θ about its saddle $\theta_{\rm cl} = \frac{2\pi}{\beta} u$
- Keep fluctuations $\varepsilon(u)$ and remove $SL(2,R)$ zero modes
- Use the symplectic form to read off the measure
- For each mode perform the Gaussian integral
- Form the infinite product over $n \geq 2$, then evaluate by zeta-regularization:

$$\prod_{n=2}^{\infty} n = \sqrt{2\pi}.$$

- At quadratic order, the zero modes are: $\varepsilon = 1, e^{\pm\frac{2\pi}{\beta}iu}$
- So we integrate over functions parametrized by

$$\varepsilon(u) = \sum_{|n|\geq 2} e^{-\frac{2\pi}{\beta}inu}\left(\varepsilon_n^{(R)} + i\varepsilon_n^{(I)}\right) \tag{30}$$

- where $\varepsilon_n^{(R)} = \varepsilon_{-n}^{(R)}$ and $\varepsilon_n^{(I)} = -\varepsilon_{-n}^{(I)}$
- The independent variables are $\varepsilon_n^{(R)}$ and $\varepsilon_n^{(I)}$ for positive $n \geq 2$
- Symplectic form in terms of these variables is:

$$\Omega = 2\alpha\frac{(2\pi)^3}{\beta^2}\sum_{n\geq 2}(n^3 - n)d\varepsilon_n^{(R)} \wedge d\varepsilon_n^{(I)} \tag{31}$$

- After also working out the action in (28) to quadratic order in ε, one finds that the properly normalized path integral is

$$e^{\frac{2\pi^2\gamma}{\beta}} \prod_{n\geq 2} 2\alpha \frac{(2\pi)^3}{\beta^2}(n^3-n) \int d\varepsilon_n^{(R)} d\varepsilon_n^{(I)} e^{-(2\pi)^4\gamma \frac{(\varepsilon_n^{(R)})^2+(\varepsilon_n^{(I)})^2}{\beta^3}(n^4-n^2)} \tag{32}$$

$$Z_{\mathsf{Sch}}^{\mathsf{disk}}(\beta) = \frac{1}{\alpha^{3/2}} \frac{\gamma^{3/2}}{(2\pi)^{1/2}\beta^{3/2}} e^{\frac{2\pi^2\gamma}{\beta}} \tag{33}$$

- We can compute the free energy given by the logarithm of the partition function:

$$-\beta F \equiv \log Z(\beta) = S_0 + \frac{2\pi^2 C}{\beta} + \frac{3}{2}\log\frac{2\pi C}{\beta} + \ldots, \tag{34}$$

- where the dots are temperature-independent
- The first two terms are classical contributions
- The third is a quantum effect. The quantum effects become large as we lower the temperature $\beta \gtrsim C$

Thank You!

Questions?