Capacity of Entanglement in Field Theory and Holography

M. Reza Mohammadi Mozaffar

University of Guilan

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دانشگاه گیلان

Ferdowsi University of Mashhad

Opening remarks

- Capacity of Entanglement:

  ✓ is a quantum information theoretic counterpart of heat capacity!

  ✓ was employed to describe topologically ordered states in CMP

  ✓ may give an accidental entanglement c-function

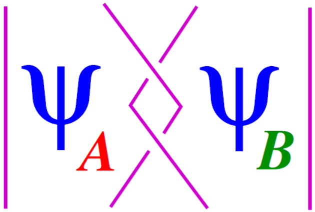
  ✓ gives some information about the possible holographic dual

  ✓ $\cdots$

Plan of the talk

1. Entanglement in QM and QFT
   - Definition and Properties
   - Entanglement Measures
2. Holographic Entanglement
3. CoE in Lifshitz QFTs
   - Lifshitz Harmonic Models (LHM)
   - Entanglement in Lifshitz Scalar Theory
   - Entanglement c-functions

Entanglement in QM and QFT



(Pure) Entangled states

- Consider two quantum systems, i.e., A and B

$$\mathcal{H}_A, \quad |i\rangle, \quad i=1,\cdots,n, \quad \mathcal{H}_B, \quad |a\rangle, \quad a=1,\cdots,m$$

- Construct M using the tensor product of A and B

$$\mathcal{H}_M=\mathcal{H}_A\otimes\mathcal{H}_B, \quad |i\rangle\otimes|a\rangle\equiv|i,a\rangle$$

- Separable states

$$|\chi\rangle_{\mathcal{H}_M}=|\psi\rangle_{\mathcal{H}_A}\otimes|\phi\rangle_{\mathcal{H}_B}$$

- (Pure) Entangled states

$$|\chi\rangle_{\mathcal{H}_M}\neq|\psi\rangle_{\mathcal{H}_A}\otimes|\phi\rangle_{\mathcal{H}_B}$$

Example: Spin 1/2 Particles

- Separable states

$$\begin{aligned} \big|\Psi_1\big\rangle &= \big|\uparrow_A\big\rangle \otimes \big|\downarrow_B\big\rangle \\ \big|\Psi_2\big\rangle &= \big|\downarrow_A\big\rangle \otimes \big|\uparrow_B\big\rangle \end{aligned}$$

- Entangled states

$$\big|\Psi_3\big\rangle = \frac{1}{\sqrt{2}}\left(\big|\uparrow_A\big\rangle \otimes \big|\downarrow_B\big\rangle \pm \big|\downarrow_A\big\rangle \otimes \big|\uparrow_B\big\rangle\right)$$

Challenge

Entanglement Measures!

Entanglement entropy, Mutual information, $\cdots$

Reduced Density Matrix

- Consider the density matrix for a pure system $M = A \cup B$

$$\rho_M = |\psi\rangle\langle\psi|$$

- Definition of reduced density matrix for A

$$\rho_A \equiv \mathrm{Tr}_B(\rho_M)$$

- For any $O_A \in A$

$$\langle O_A \rangle = \mathrm{Tr}_A(\rho_A O_A)$$

Example: Spin 1/2 Particles

- Separable states

$$|\Psi_1\rangle = |\uparrow_A \downarrow_B\rangle \rightarrow \rho_A^{(1)} = |\uparrow_A\rangle\langle\uparrow_A| \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$|\Psi_2\rangle = |\downarrow_A \uparrow_B\rangle \rightarrow \rho_A^{(2)} = |\downarrow_A\rangle\langle\downarrow_A| \sim \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

- Entangled states

$$|\Psi_3\rangle = \frac{|\Psi_1\rangle \pm |\Psi_2\rangle}{\sqrt{2}} \rightarrow \rho_A = \frac{\rho_A^{(1)} + \rho_A^{(2)}}{2} \sim \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Entanglement Entropy (EE)

- von-Neumann entropy for ρ_A

$$S_A \equiv -\mathrm{Tr}_A \left(\rho_A \log \rho_A\right)$$

- Example:

$$\left|\Psi_1\right\rangle \text{ and } \left|\Psi_2\right\rangle, \qquad S_A = 0$$

$$\left|\Psi_3\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\Psi_1\right\rangle \pm \left|\Psi_2\right\rangle\right), \qquad S_A = \log 2$$

Entanglement Entropy (EE)

- Properties

1. EE corresponds to a non-linear operator in QM.
2. When $A \cup B$ is pure $$S(A) = S(B)$$
3. Subadditivity $$S(A) + S(B) \geq S(A \cup B)$$
4. Strong subadditivity $$S(A) + S(B) \geq S(A \cup B) + S(A \cap B)$$

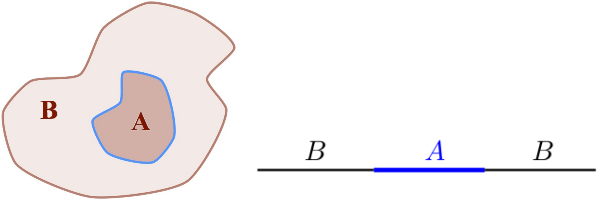
and $\cdots$

Challenge

Generalization to QFT (Continuum Limit)?

Geometric entropy

- Consider a local d-dimensional QFT on $\mathbb{R} \times \mathcal{M}^{(d-1)}$
- Divide $\mathcal{M}^{(d-1)}$ into two parts



- Locality implies $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$$S_A = -\text{Tr}_A \left(\rho_A \log \rho_A\right)$$

Properties of EE in QFTs

- Infinite number of DOFs in QFT → Divergent EE
- In $d-1$ spatial dimensions we should have for any QFT

$$S_A \propto \frac{\mathcal{S}_{d-2}}{\epsilon^{d-2}} + \cdots + \frac{\mathcal{S}_1}{\epsilon} + \mathcal{S}_{\text{univ.}} \log \epsilon + \mathcal{S}_0$$

ϵ : Lattice constant (UV cutoff^{-1})

$\mathcal{S}_{d-2}$: Area of the entangling surface

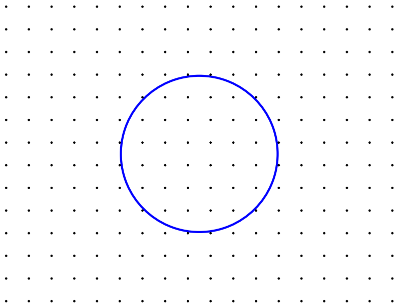
$\mathcal{S}_{\text{univ.}}$: Universal coefficient

$\mathcal{S}_0$: Finite part

Area Law

$$S_A \propto \frac{\mathcal{S}_{d-2}}{\epsilon^{d-2}} + \cdots$$

- ϵ : Lattice constant (UV cutoff^{-1})
- $\mathcal{S}_{d-2}$: Area of the entangling surface



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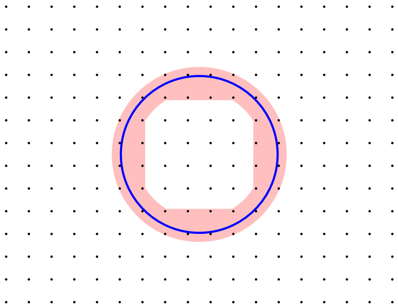


Replica trick

$$S_A = -\mathrm{Tr}_A \left(\rho_A \log \rho_A\right)$$

- Taking the logarithm of ρ_A is very complicated!
- Renyi Entropy

$$S_A^{(n)} = \frac{1}{1-n} \log \mathrm{Tr}_A \left(\rho_A^n\right)$$

- Replica Trick

$$S_A = \lim_{n \to 1} S_A^{(n)}$$

[Bombelli-Koul-Lee-Sorkin ’86, Callan-Wilczek ’94]

✓ Modular Hamiltonian (K_A)

- ρ_A is both hermitian and positive semidefinite

$$\rho_A = e^{-K_A} \qquad\qquad K_A: \text{ Modular Hamiltonian}$$

(Similar to $\rho = e^{-\beta H}$!)

- Generically K_A is not a local operator!
- The expectation value of K_A:

$$S_A = \langle K_A \rangle$$

✓ Capacity of Entanglement (Modular Fluctuations)

- The variance of K_A:

$$C_A = \langle K_A^2 \rangle - \langle K_A \rangle^2$$

- C_A contains information about the width of the eigenvalue distribution of ρ_A

$$C_A \equiv \lim_{n\to 1} C_n = \lim_{n\to 1} n^2 \frac{\partial^2}{\partial n^2}\left((1-n)S_A^{(n)}\right)$$

H. Yao and X. L. Qi, Phys. Rev. Lett. (2010)

- Modular Entropy $\tilde{S}_A \equiv n^2 \frac{\partial}{\partial n}\left(\frac{n-1}{n} S_A^{(n)}\right)$

$$C_n = -\frac{\partial \tilde{S}_A}{\partial n}$$

✓ Capacity of Entanglement C_A

Statistical Mechanics	Quantum Information
Inverse Temperature $T^{-1} \equiv \beta$	Replica Index $n \equiv \beta$
Hamiltonian H	Modular Hamiltonian K_A
Part. Func. $Z = \mathrm{Tr}\, e^{-\beta H}$	Replica Part. Func. $Z = \mathrm{Tr}\, e^{-nK_A}$
Energy $E = -\frac{\partial}{\partial \beta} \log Z$	Replica Energy $E = -\frac{\partial}{\partial n} \log Z$
Free Energy $F = \frac{-1}{\beta} \log Z$	Replica Free Energy $F = \frac{-1}{n} \log Z$
Thermal Entropy $S_{\mathrm{th.}} = \beta^2 \frac{\partial F}{\partial \beta}$	Modular Entropy $\tilde{S}_A = n^2 \frac{\partial F}{\partial n}$
Thermal Capacity $C_{\mathrm{th.}} = -\beta \frac{\partial S_{\mathrm{th.}}}{\partial \beta}$	CoE $C_A = -n \frac{\partial \tilde{S}_A}{\partial n}$
...	...

✓ Capacity of Entanglement C_A

Renyi entropy inequalities ↔ Stability of the thermal system

$$\left\{\begin{array}{l} \frac{\partial}{\partial n}\left(\frac{n-1}{n}S_A^{(n)}\right) \geq 0 \\ \frac{\partial}{\partial n}\left((n-1)S_A^{(n)}\right) \geq 0 \\ \frac{\partial^2}{\partial n^2}\left((n-1)S_A^{(n)}\right) \geq 0 \end{array}\right. \quad \leftrightarrow \quad \left\{\begin{array}{l} \tilde{S}_A(n) \geq 0 \\ E(n) \geq 0 \\ C_A(n) \geq 0 \end{array}\right.$$

[Nakaguchi & Nishioka, JHEP 2016]

✓ Capacity of Entanglement C_A

- For finite dimensional Hilbert spaces ($\mathrm{Dim}\mathcal{H} = D$):

$$S_A \leq S_{\text{max.}} = \log N \qquad\qquad C_A \leq C_{\text{max.}} \sim \frac{S_{\text{max.}}^2}{4}$$

- For maximally mixed states ($\rho = \frac{1}{D}\mathbb{I}$):

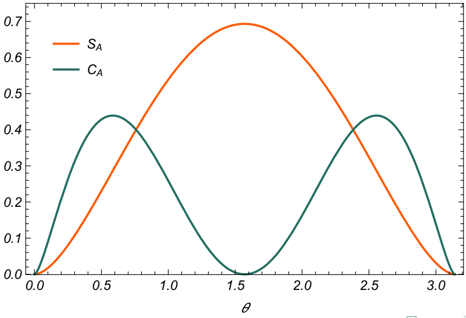
$$S_A = \log D \qquad\qquad C_A = 0$$

$$C_A \ll S_A$$

✓ Capacity of Entanglement C_A

- Two-qubit systems:

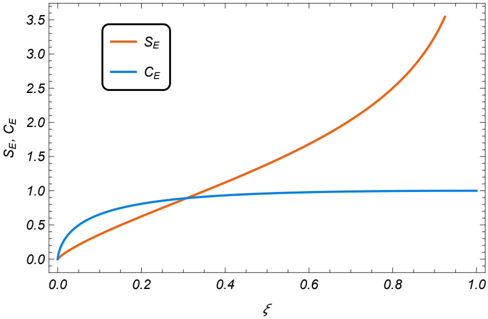
$$|\psi\rangle = \cos\frac{\theta}{2}|10\rangle + e^{i\phi}\sin\frac{\theta}{2}|01\rangle$$



✓ Capacity of Entanglement C_A

- Two coupled harmonic oscillators in vacuum state:

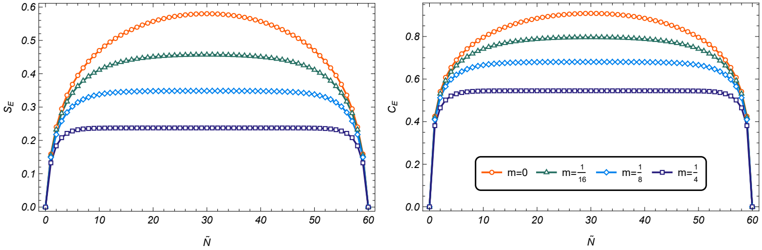
$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{k_0}{2}(x_1^2 + x_2^2) + \frac{k_1}{2}(x_1 - x_2)^2, \qquad \xi = \xi\left(\frac{k_1}{k_0}\right)$$



- $\frac{C_A}{S_A} \ll 1$ in $\xi \to 1$ corresponds to strong coupling limit

✓ Capacity of Entanglement C_A

- A lattice of oscillators in vacuum state:



✓ Capacity of Entanglement C_A

- In $2d$ CFTs: static states, single interval

$$C_A = S_A = \frac{c}{6} W_A + \mathcal{O}(1)$$

$$W_A = \begin{cases} 2\log\frac{\ell}{\epsilon}, & T = 0, L \to \infty \\ 2\log\left(\frac{L}{\pi\epsilon}\sin\frac{\pi\ell}{L}\right), & T = 0, L \neq \infty \\ 2\log\left(\frac{\beta}{\pi\epsilon}\sin\frac{\pi\ell}{\beta}\right), & T \neq 0, L \neq \infty \end{cases}$$

✓ Capacity of Entanglement C_A

- In $2d$ Free Massive Models: static states, single interval
- Real Scalars

$$
\begin{aligned}
S_A &= \frac{c}{3}\log\frac{\ell}{\epsilon} + \frac{c}{3}\log\frac{\log m\ell}{\log m\epsilon} + \cdots \\
C_A &= \frac{c}{3}\log\frac{\ell}{\epsilon} + \cdots
\end{aligned}
$$

- Dirac Fermions

$$
C_A = S_A = \frac{c}{3}\log\frac{\ell}{\epsilon} - \frac{c}{6}\left(m\ell\log m\ell\right)^2 + \cdots
$$

Challenges

1. Extension to higher dimensions and general shapes
2. Investigating the role of symmetry
3. Turning on non-trivial interactions
4. . . .

Holography may help us to overcome these problems

Holographic Entanglement

Maldacena's Conjecture

- Strongly coupled CFT_d ∼ Classical gravity in AAdS_{d+1}

$$\log \mathcal{Z}_{\mathrm{CFT}_d} \sim \log \mathcal{Z}_{\mathrm{AdS}_{d+1}}$$

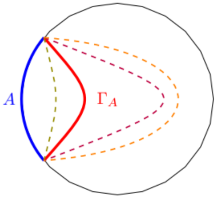
- Dictionary

CFT_d	AdS_{d+1}
$\mathcal{O}, \mathcal{J}_\mu, \mathcal{T}_{\mu\nu}$	$\phi, A_\mu, g_{\mu\nu}$
Energy scale	Radial coordinate
$T \neq 0,\ S \neq 0$	Black-hole
$\dots$	$\dots$
ρ_A	?
S_A	?
C_A	?

Gravity Dual of Entanglement Entropy

- Einstein Gravity: $I_{\text{bulk}} = I_{\text{EH}} + I_{\Lambda} + I_{\text{matters}}$

$$S_A = \min \left(\frac{\text{Area}(\Gamma_A)}{4G_N} \right)$$



- Properties of the RT surface:
  1. $\partial\Gamma_A = \partial A$
  2. Γ_A is a codimension-2 spacelike hypersurface
  3. Γ_A is homologous to A (homology constraint)

[Ryu & Takayanagi, PRL 2006]

Gravity Dual of Modular Entropy

- Einstein Gravity: $I_{\text{bulk}} = I_{\text{EH}} + I_{\Lambda} + I_{\text{matters}} + I_{\text{brane}}$

$$I_{\text{brane}} = T_n \mathcal{A}_{\text{brane}} \qquad T_n = \frac{1}{4G_N} \frac{n-1}{n}$$

$$\tilde{S}_A = \min \left(\frac{\text{Area}(\tilde{\Gamma}_A)}{4G_N} \right)$$

- n quantifies the strength of the backreaction!
- Properties of the Dong surface:
  1. $\partial \tilde{\Gamma}_A = \partial A$
  2. $\cdots$

[Dong, Nature Communications 2016]

Gravity Dual of C_E

- Einstein Gravity: $I_{\text{bulk}} = I_{\text{EH}} + I_{\Lambda} + I_{\text{matters}} + I_{\text{brane}}$

$$C_A = \frac{1}{64G_N^2} \int d^{d-1}\sigma \sqrt{h(\sigma)} \int d^{d-1}\sigma' \sqrt{h(\sigma')} h^{ij} G_{ij;kl} h^{kl}$$

- h_{ij}: induced metric on the brane
- $G_{ij;kl}$: graviton propagator

[Nakaguchi & Nishioka, JHEP 2016]

Gravity Dual of CoE

- In holographic duals of Einstein gravity: $C_A = S_A$
- In Gauss-Bonnet gravity ($d = 4$): $C_A \propto c, \qquad S_A \propto a$

Further directions

Some achievements/extensions related to CoE

- C_E as a diagnostic of the quantum phase transitions
- C_E as an (accidental) entropic c-function
- C_E and quench dynamics
- C_E in excited states
- C_E in non-local theories
- Symmetry-resolved CoE
- C_E in non-relativistic theories
- Flat entanglement spectrum $\leftrightarrow$ fixed-area states
- $\cdots$

CoE in Lifshitz QFTs

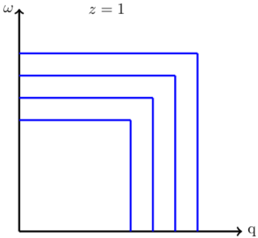
Lifshitz symmetry

- Anisotropic scaling invariance

  $t \to \lambda^z t, \qquad \vec{x} \to \lambda \vec{x}, \qquad z: \quad \text{Dynamical critical exponent}$

  [E. M. Lifshitz 1941]

- Different scaling dimension $\longrightarrow$ Different RG rate



[J. A. Hertz 1976]

Free Massless Scalar Theory

Lorentz vs. Lifshitz		
	Lorentz	Lifshitz
Lagrangian	$\frac{1}{2}\left(\dot{\phi}^2 - (\partial_i \phi)^2\right)$	$\frac{1}{2}\left(\dot{\phi}^2 - (\partial_i^z \phi)^2\right)$
Mass Dimensions	$[t] = -1,\ [\phi] = \frac{d-1}{2}$	$[t] = -z,\ [\phi] = \frac{d-z}{2}$
Dispersion Relation	$\omega = k$	$\omega = k^z$
Group Velocity	$v_g = 1$	$v_g = z\, k^{z-1}$

Lifshitz-type QFT on a (1+1)d Square Lattice

✓ Harmonic lattice:

- Free Scalar Theory

$$H = \frac{1}{2}\int dx \left[\dot{\phi}^2 + (\partial\phi)^2 + m^2\phi^2\right]$$

- System of N Harmonic Oscillators

$$H = \sum_{n=0}^{N} \left[\frac{1}{2}p_n^2 + \frac{1}{2}\left(q_n - q_{n-1}\right)^2 + \frac{m^2}{2}q_n^2\right]$$

Nearest Neighbor Interaction

- Dispersion Relation

$$\omega_k = \sqrt{m^2 + k^2} \quad \longrightarrow \quad \omega_k = \sqrt{m^2 + (2\sin\frac{\pi k}{N})^2}$$

Lifshitz-type QFT on a Square Lattice

✓ Lifshitz harmonic lattice:

- Lifshitz-type scalar theory

$$H = \frac{1}{2}\int dx \left[\dot{\phi}^2 + (\partial^z \phi)^2 + m^{2z}\phi^2\right]$$

- Discretization on a Square Lattice

$$H = \sum_{n=1}^{N} \left[\frac{p_n^2}{2} + \frac{1}{2}\left(\sum_{k=0}^{z}(-1)^{z+k}\binom{z}{k} q_{n-1+k}\right)^2 + \frac{m^{2z}}{2} q_n^2\right]$$

Long Range Interaction (depending on z)

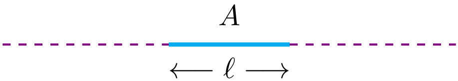
- Dispersion Relation

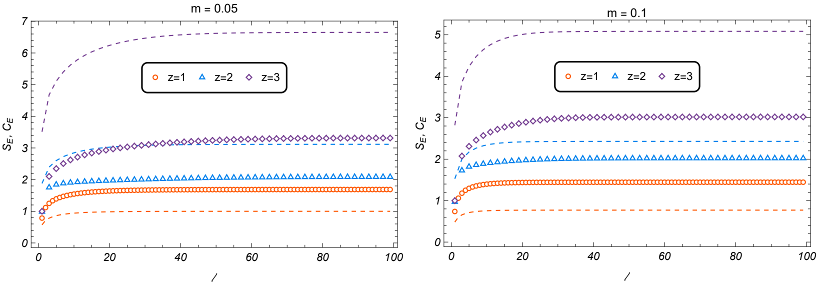
$$\omega_k = \sqrt{m^{2z} + k^{2z}} \quad \longrightarrow \quad \omega_k = \sqrt{m^{2z} + (2\sin\frac{\pi k}{N})^{2z}}$$

[MM, Mollabashi 1705.00483-1712.03731; He, Magan and Vandoren, 1705.01147]

Entanglement in Lifshitz Scalar Theory

CoE for Massive Scalar (1 + 1-dimensions)





[Khoshdooni, Babaei, MM, 2025]

✓ Measures increases while z is increased!

[MM, Mollabashi 1705.00483-1712.03731; He, et.al., 1705.01147]

Results

- Kinetic Term in Hamiltonian $(\partial_i^z \phi)^2$
- Discretize Hamiltonian on a Lattice

$z = 1$ $\{\phi_{i+1}, \phi_i, \phi_{i-1}\} \in \mathcal{H}$

$z = 2$ $\{\phi_{i+2}, \phi_{i+1}, \phi_i, \phi_{i-1}, \phi_{i-2}\} \in \mathcal{H}$

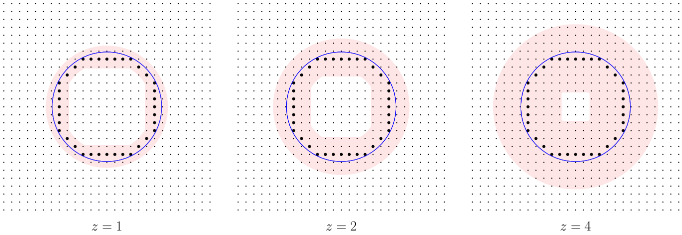
.

.

z $\{\phi_{i+z}, \phi_{i+z-1}, \phi_{i+z-2}, \cdots, \phi_{i-z+2}, \phi_{i-z+1}, \phi_{i-z}\} \in \mathcal{H}$

Results

- For larger values of z the number of correlated points due to the kinetic term increases



- The correlation between points inside and outside the entangling region increases

Results

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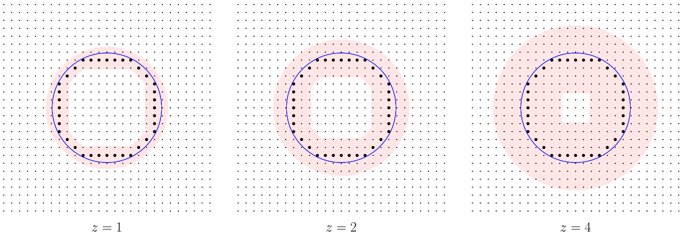
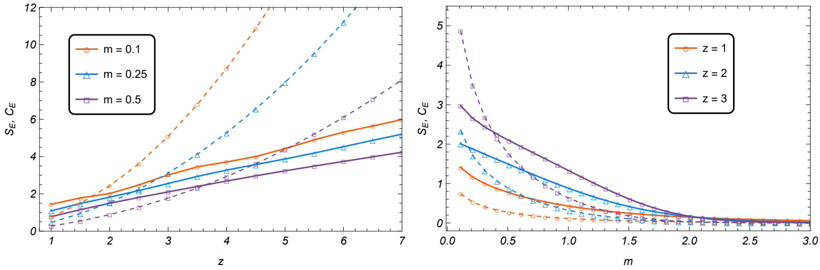


- The correlation between points inside and outside the entangling region increases

Non-local effects due to the nontrivial z

CoE for Massive Scalar (1 + 1-dimensions)

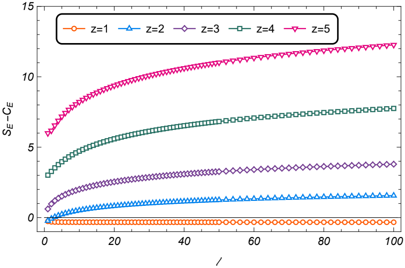


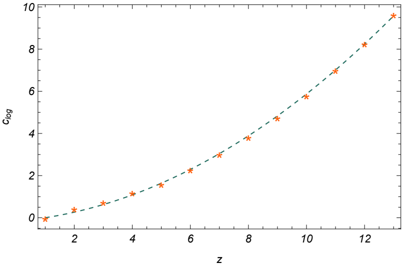
✓ $z \gg 1 \;\rightarrow\; C_E \ll S_E$: ρ_A becomes maximally mixed!

[Khoshdooni, Babaei, MM, 2025]

CoE for Massless Scalar (1 + 1-dimensions)

✓ Dirichlet boundary condition ($m = 0$)





✓ $$S_E - C_E \sim c_{\log} \log \frac{\ell}{\epsilon} + c_0 \qquad c_{\log} \propto (z-1)$$

[Khoshdooni, Babaei, MM, 2025]

Zamolodchikov's c-theorem in $2d$

✓ $\exists\, C(g_i) > 0$ such that:

- C is monotonically decreasing under RG flow

$$\frac{\partial C}{\partial M} < 0$$

- fixed points of the flow are critical points of C

$$\frac{\partial C}{\partial g_i}\Big|_{g_i = g_i^*} = 0$$

- the fixed point value of this function is the central charge

$$C(g_i^*) = c$$

[A.B. Zamolodchikov 1986]

Casini and Huerta's (entropic) c-theorem

- Strong subadditivity + Lorentz invariance (in $2d$):

$$C(\ell) = \ell \frac{\partial S(\ell)}{\partial \ell}$$

$$C'(\ell) = \ell \frac{\partial^2 S(\ell)}{\partial \ell^2} + \frac{\partial S(\ell)}{\partial \ell} < 0$$

✓ $C_{\text{CFT}}(\ell) = \frac{c}{3}$ $\quad S(\ell) = \frac{c}{3} \log \frac{\ell}{\epsilon}$

- Similar approach in $3d$ gives the F-theorem $\quad S(\ell) = \alpha \frac{\ell}{\epsilon} - F$

[Casini & Huerta 2004, 2012; Liu & Mezei 2012]

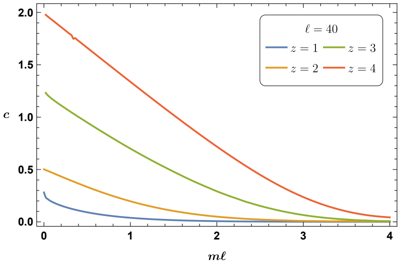
- F-theorem can also derived in terms of mutual information

[Casini, Huerta & Myers 2015]

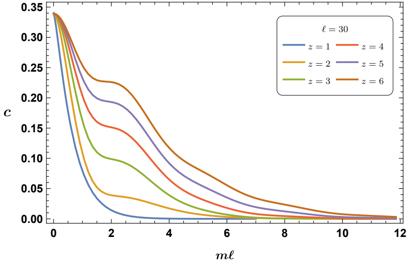
Non-relativistic RG flow

- Non-relativistic c-theorem!

Boson



Fermion



[Vasli, Babaei, MM & Mollabashi 2024]

CoE and Non-relativistic RG flow

- Other entanglement based c functions!

$$c_S = \ell \frac{\partial S_A}{\partial \ell} \qquad c_C = \ell \frac{\partial C_A}{\partial \ell} \qquad c_M = \ell \frac{\partial M_A}{\partial \ell}$$

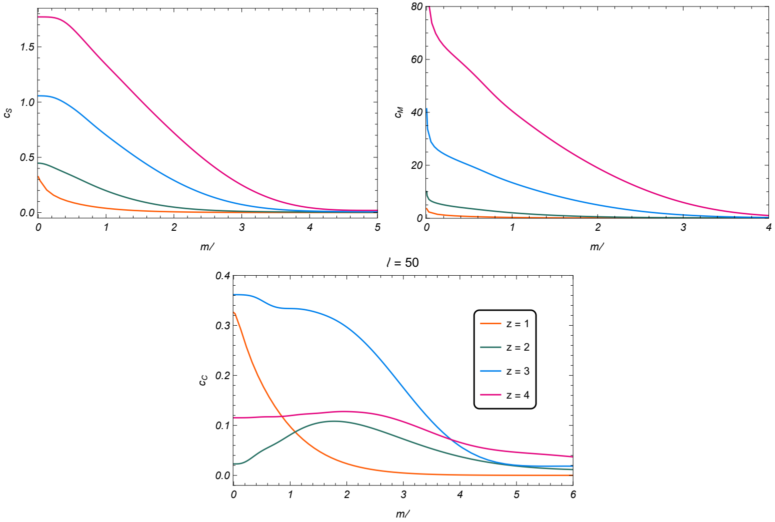
$$\checkmark\, M_A = \langle (1+K_A)^2 \rangle = C_A + (S_A + 1)^2$$

- Majorization & Schur concave measures

$$\rho \succ \sigma \;\longrightarrow\; \begin{cases} S(\rho) \leq S(\sigma), \\ M(\rho) \leq M(\sigma), \\ C(\rho) \;\; ? \;\; C(\sigma), \end{cases}$$

[Boes, Ng & Wilming, PRX Quantum (2022)]

Entanglement Measures and Non-relativistic RG flow



Entanglement Measures and Non-relativistic RG flow

Scalar Field			
	S_A	C_A	M_A
SSA	✓	×	×
Schur concavity	✓	×	✓
Monotonic c-function $(z=1)$	✓	✓	✓
Monotonic c-function $(z>1)$	✓	×	✓

Further directions

- C_E as a diagnostic of the quantum phase transitions
- C_E and quench dynamics
- C_E in excited states
- Symmetry-resolved CoE
- Flat entanglement spectrum $\leftrightarrow$ fixed-area states
- Magical QFTs and non-flatness of entanglement spectrum
- $\dots$

Some References

- Y. Nakaguchi and T. Nishioka, JHEP 12, 129 (2016).
- J. De Boer, J. Jarvela and E. Keski-Vakkuri, Phys. Rev. D 99, no.6, 066012 (2019).
- R. Arias, G. Di Giulio, E. Keski-Vakkuri and E. Tonni, JHEP 03, 175 (2023).
- M. R. Mohammadi Mozaffar and A. Mollabashi, JHEP 07, 120 (2017) .
- M. J. Vasli, K. Babaei Velni, M. R. Mohammadi Mozaffar and A. Mollabashi, JHEP 09, 122 (2024).

HEE

Backup slides

Different Notions of EE in QFT

- Different Hilbert space decompositions lead to different types of EE

  1. Field space entanglement entropy

     [Yamazaki '13]

  2. Momentum space entanglement entropy

     [Balasubramanian-McDermott-Van Raamsdonk '11]

  3. Geometric (Entanglement) entropy

     [Bombelli-Koul-Lee-Sorkin '86, Srednicki '93, Callan-Wilczek '94]

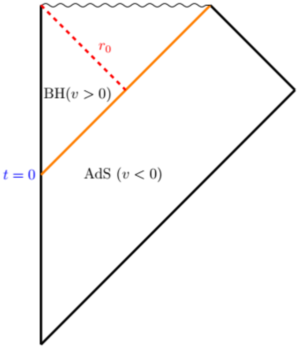
HRT Proposal

- Properties of the co-dimension two HRT surface:
  1. $\partial\Gamma_A = \partial A$
  2. A surface with vanishing expansions of null geodesics
  3. A saddle point of the proper area functional
- Some properties of HRT prescription:
  1. Obeys strong subadditivity of HEE
  2. Obeys monogamy of mutual information

Time evolution of HEE in quenched holographic systems!

HRT Proposal

- Quench process corresponds to black hole formation
- Vaidya geometry describes the collapse of a shell of matter



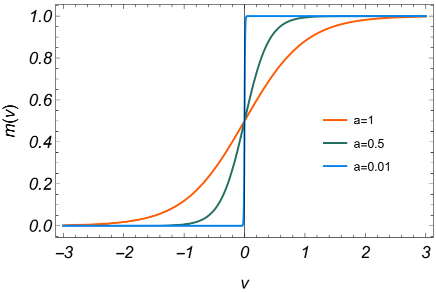
- Early time: Pure AdS Late time: AdS Black-brane

HRT Proposal

- A holographic set-up dual to a global quench

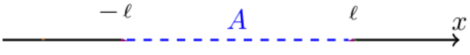
$$ds^2 = \frac{L^2}{r^2}\Big(- f(r,v)dv^2 - 2drdv + d\vec{x}^2 \Big), \qquad dv = dt - \frac{dr}{f}$$

$$f(r,v) = 1 - m(v)r^d \qquad\qquad m(v) = \frac{m}{2}\left(1 + \tanh\frac{v}{a}\right)$$



HRT Proposal

- Example Quantum quench in CFT_2



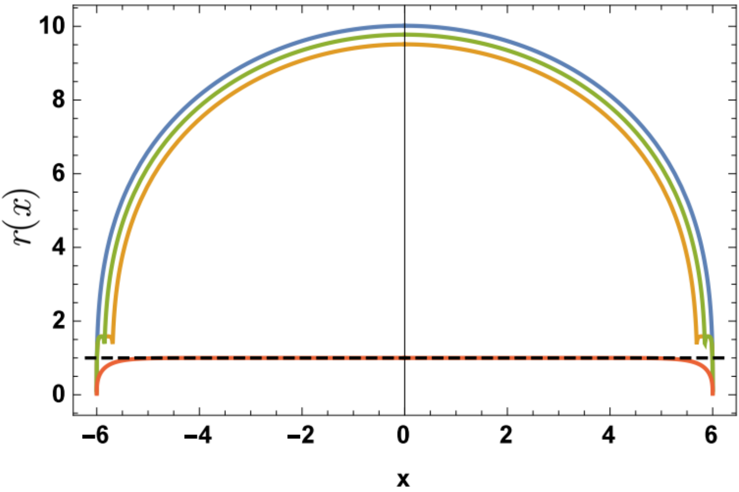
- profile of the HRT surface: $r = r(x)$ and $v = v(x)$
- Area functional

$$\mathcal{A} = \int \frac{dx}{r} \sqrt{1 - 2v'r' - v'^2 f(r,v)\, dr}$$

- Boundary conditions $\quad r\left(\pm\frac{\ell}{2}\right) = 0 \qquad v\left(\pm\frac{\ell}{2}\right) = t$
- Exercise

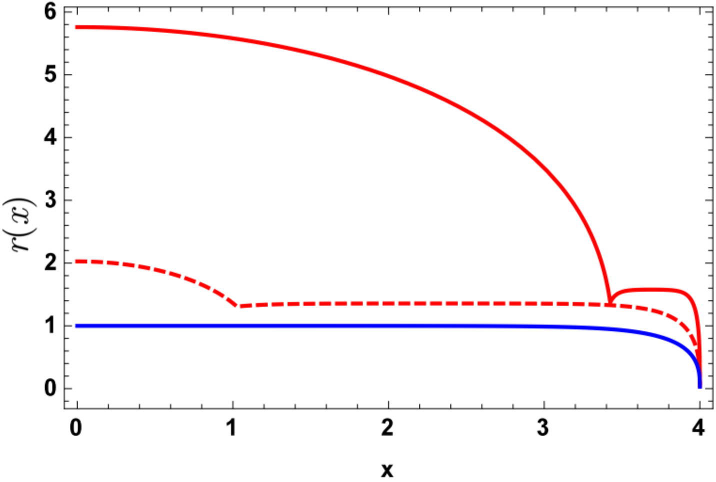
  Find the profile of the HRT surface and compute $S_A(t)$

HRT Proposal



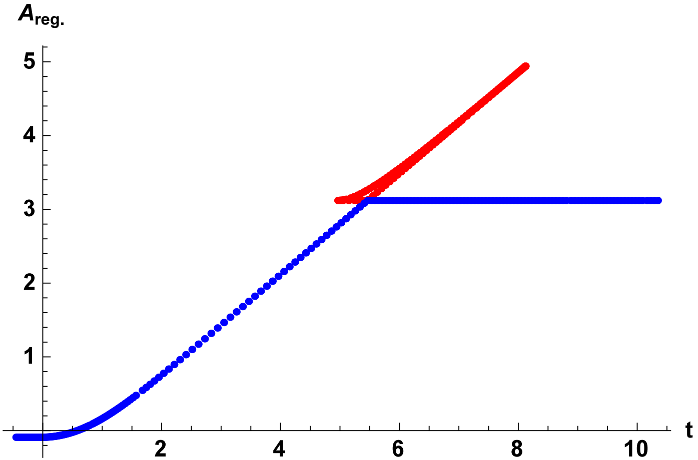
- An evolution from AdS profile to AdS Black-brane profile

HRT Proposal



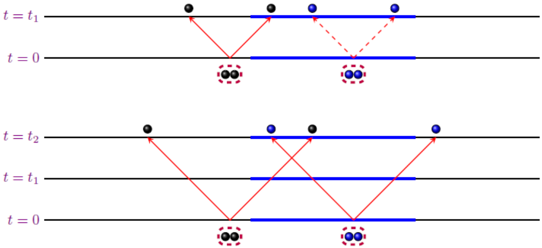
- Presence of multiple solutions to EoMs for a given time!

HRT Proposal



HEE

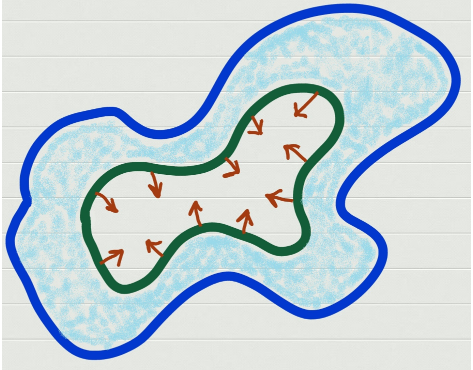
- According to HRT prescription (after a quench) HEE has:
  1. an early time quadratic growth
  2. an intermediate linear growth
  3. a late time saturation
- Quasi particle picture!



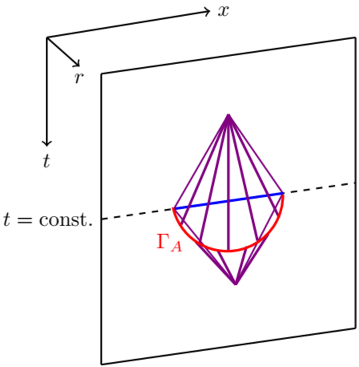
HEE

- Entanglement Tsunami

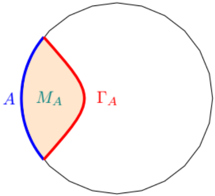
A picture for the growth of EE in a strongly coupled QFT!



Entanglement Wedge



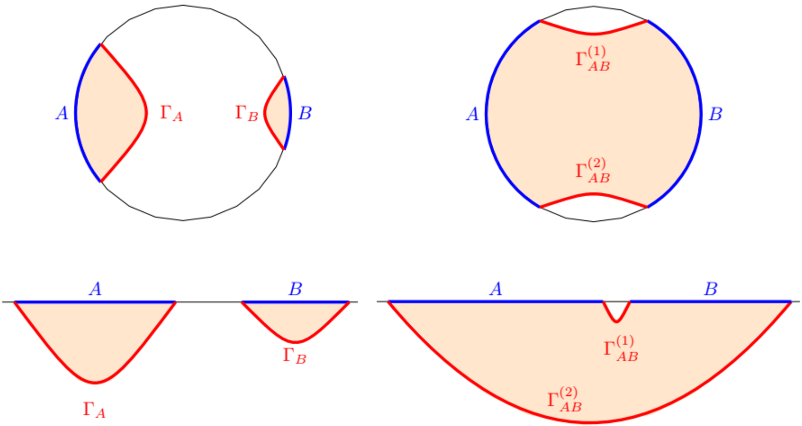
Time Slice of EW



- $\partial M_A = A \cup \Gamma_A$

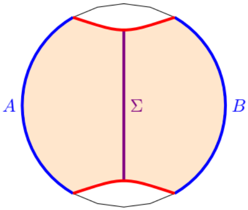
Entanglement Wedge

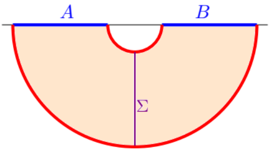
- Two boundary subsystems: $\partial M_{AB} = A \cup B \cup \Gamma_{AB}$



Entanglement Wedge Cross-section E_W

$$E_W = \frac{\text{area}\,(\Sigma)}{4G_N}$$





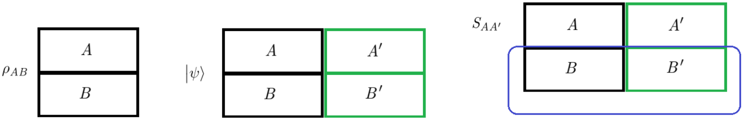
- A Holographic Dual for
  1. E_P [Takayanagi, Umemoto '17, Nguyen et. al. '17]
  2. $\mathcal{E}$ [Kudler-Flam, Ryu'18]
  3. S_O [Tamaoka'18]
  4. S_R [Dutta, Faulkner, '19]

Entanglement of Purification

- Consider a mixed state in $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ describe by ρ_{AB}
- Enlarge $\mathcal{H}$ to $\mathcal{H}_{AA'} \otimes \mathcal{H}_{BB'}$ by adding some auxiliary degrees of freedom $\rho_{AB} = \mathrm{Tr}_{A'B'}|\psi\rangle\langle\psi|$

  Purification is not unique!

$$E_P = \min_{\rho_{AB}=\mathrm{Tr}_{A'B'}|\psi\rangle\langle\psi|} S_{AA'}$$

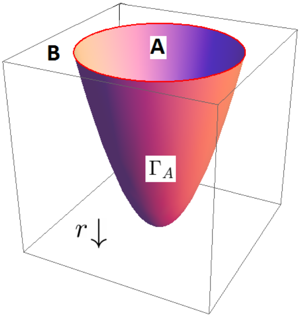


Entanglement of Purification

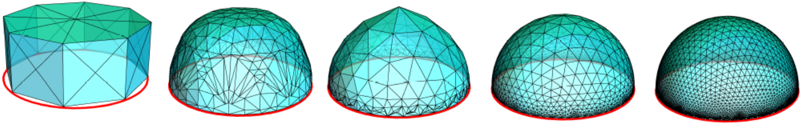
E_P:

- reduces to S_A for pure states
- enjoys several inequalities
  1. $E_P(A,B) \leq E_P(A, B \cup C)$
  2. $\frac{I(A,B)}{2} \leq E_P(A,B) \leq \min\{S_A, S_B\}$
  3. $\frac{I(A,B)+I(A,C)}{2} \leq E_P(A, B \cup C)$
- is a UV finite quantity
- $\cdots$

Ryu-Takayanagi Proposal

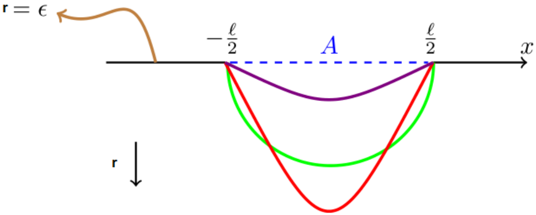


- Choosing the minimal surface



[Fonda et.al. '14]

Example: AdS_3/CFT_2



$$S_A = \frac{c}{3} \log \frac{\ell}{\epsilon}$$

- [Ryu & Takayanagi, 2006]

Ryu-Takayanagi Proposal

- Further generalizations:

  higher dimensions, mixed and excited states, entanglement inequalities, information and entanglement measures, multipartite entanglement, time evolution $\cdots$

Holographic Proposals

Some achievements/extensions related to HEE

- Holographic entanglement measures
- Entanglement and renormalization
- Entropic c-functions
- Surface/State correspondence, AdS/cMERA
- HEE & causality in CFT and gravity
- Entanglement inequalities holographic entropy Cone
- Higher dimensional twist operators
- Holographic quantum quench
- Geometry from entanglement
- …

✓ Some basic questions

- What is the gravity dual of ρ_A?
- What part of the bulk can be fully reconstructed from ρ_A?
- Given ρ_A, in what region of the bulk can we uniquely reconstruct the geometric data $(g_{\mu\nu})$?
- Can we find gravitational dynamics from entanglement pattern?

The bulk reconstruction program!

Linearized gravitational dynamics

1. Consider perturbations around the AdS geometry

$$g_{\mu\nu} = g_{\mu\nu}^{\text{AdS}} + h_{\mu\nu}, \qquad \mathcal{O}(h) \ll \mathcal{O}(g)$$

2. Choose a ball-shaped entangling region

- Linearized gravity from the 1st law

$$\delta\langle S\rangle = \delta\langle K_\sigma\rangle \Leftrightarrow \delta\mathbf{E} = 0$$

Linearized Einstein equations $\leftrightarrow$ Entanglement 1st law

Some achievements/extensions related to HEE

- Holographic computational complexity
- Holographic tensor network and quantum error correction
- Resolving information paradox
- Islands and replica wormholes
- dS/CFT and Timelike EE
- $\cdots$