Ricci Flow Equations from $T\bar{T}$ Deformations

Presented by:

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Based on:

arxiv:2202.11156, , and 2503.xxxx

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Advanced Topics in Quantum Gravity
Ferdowsi University of Mashhad
February 5th, 2025

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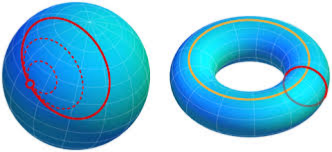
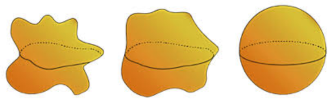
The Poincaré conjecture (in topology):



The Ricci flow equation, introduced by Richard Hamilton as:

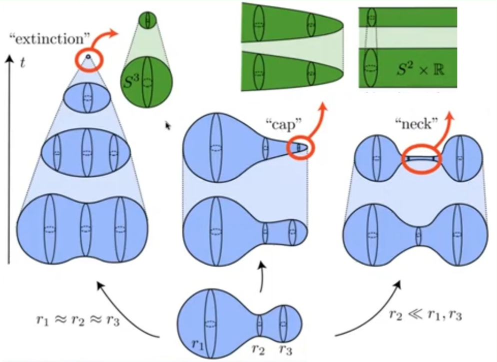
$$\frac{d\, g_{\mu\nu}(\lambda)}{d\,\lambda} = -2R_{\mu\nu}$$



Grigori Perelman proof:

On November 11, 2002, Russian mathematician Grigori Perelman posted the first of a series of three eprints on arXiv outlining a solution of the Poincaré conjecture. Perelman's proof uses a modified version of a Ricci flow program developed by Richard S. Hamilton.

Grigori Perelman proof:



Theorem (Perelman 2002, Brendle 2018)

The only possible 3D singularity models are:

- spherical: S^3/Γ
- cylindrical: $S^2 \times \mathbb{R}/\Gamma$
- Bryant soliton

History of Ricci Flow Equations

Ricci-Bourguignon flows:

A family of metrics $g_{\mu\nu}$ on an d-dimensional Riemannian manifold $(M^d, g_{\mu\nu})$ evolves according to the Ricci-Bourguignon flow if $g_{\mu\nu}$ satisfies the following evolution equation:

$$\frac{d\, g_{\mu\nu}}{d\, \lambda} = -2(\, R_{\mu\nu} - \rho\; R\, g_{\mu\nu}) \,,$$

Special cases of Ricci-Bourguignon flows :

- Ricci flow: It is well known that for $\rho = 0$, we obtain the Ricci flow equation as : $\frac{d\, g_{\mu\nu}}{d\, \lambda} = -2R_{\mu\nu}$.
- Einstein Ricci flow: The Einstein tensor, denoted as $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\; R\, g_{\mu\nu}$, leads to the Einstein flow equation for values $\rho = \frac{1}{2}$ as follows: $\frac{d\, g_{\mu\nu}}{d\, \lambda} = -2G_{\mu\nu}$,
- Traceless Ricci flow: If we consider $\rho = \frac{1}{d}$, we will have a traceless Ricci flow as follows: $\frac{d\, g_{\mu\nu}}{d\, \lambda} = -2(\, R_{\mu\nu} - \frac{1}{d}\, R\, g_{\mu\nu})$.
- Schouten Ricci flow: For $\rho = \frac{1}{2(d-1)}$, we have the Schouten Ricci flow equation as $\frac{d\, g_{\mu\nu}}{d\, \lambda} = -2(\, R_{\mu\nu} - \frac{1}{2(d-1)}\, R\, g_{\mu\nu})$.

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Metric approach

In [2206.03415 [hep-th]], the authors examine a family of Lorentzian manifolds $(M^d, g_{\mu\nu})$ that depend on a parameter λ:

$$\mathcal{S}(\lambda)_{\lambda+\delta\lambda} = \mathcal{S}(g_{\mu\nu})_\lambda + \delta\lambda \int d^dX \sqrt{g}\, \mathcal{O}(g_{\mu\nu})_\lambda^{[d,r]}$$

Let $\delta g_{\mu\nu} = \delta\lambda\, h_{\mu\nu}$ represent an infinitesimal deformation of the metric, with $h_{\mu\nu}$.

Taylor expand:

$$\begin{aligned}\hat{\mathcal{S}}(h_{\mu\nu}) &= \mathcal{S}_\lambda(g_{0\mu\nu} + \delta\lambda\, h_{\mu\nu}) + \int d^dX\sqrt{g}\left[c\left(h_{\alpha\beta}h^{\alpha\beta} - \frac{r}{dr-1}h_\alpha{}^\alpha h^\beta{}_\beta\right)\right]\delta\lambda \\ &\sim \mathcal{S}_\lambda(g_{0\mu\nu}) + \int d^dX\sqrt{g}\left[c\big(h_{\alpha\beta}h^{\alpha\beta} - \frac{r}{dr-1}h_\alpha{}^\alpha h^\beta{}_\beta\big) - \frac{1}{2}h_{\alpha\beta}T^{\alpha\beta}\right]\delta\lambda\end{aligned}$$

Critical point:

$$\mathcal{T}_{\mu\nu} = \left.\frac{\delta\hat{\mathcal{S}}(h_{\mu\nu})}{\delta h^{\mu\nu}}\right|_{h=\tilde{h}} = 2\,c\left(\tilde{h}_{\mu\nu} - \frac{r}{dr-1}\tilde{h}^\alpha{}_\alpha g_{\mu\nu}\right) - \frac{1}{2}T_{\mu\nu} = 0,$$

Metric approach

We can determine the value of $\tilde{h}_{\mu\nu}$ as follows:

$$\tilde{h}_{\mu\nu} = \frac{1}{4\,c}(T_{\mu\nu} - rT^{\alpha}{}_{\alpha}\, g_{\mu\nu})$$

R. Conti, J. Romano, and R. Tateo, [arXiv:2206.03415].

Ricci flows:

- Einstein field equations:

$$\tilde{G}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2}\,\tilde{R}\,\tilde{g}_{\mu\nu} = T_{\mu\nu}$$

- Analogous Ricci-Bourguignon flow equation:

$$\frac{d\,\tilde{g}_{\mu\nu}}{d\,\lambda} = \frac{1}{4\,c}(\,\tilde{R}_{\mu\nu} - (r - \frac{d\,r}{2} + \frac{1}{2})\,\tilde{R}\,\tilde{g}_{\mu\nu})$$

Einstein says go outside!!!!!!!!!!

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}} = \frac{-2}{\sqrt{-g}} \frac{\partial \sqrt{-g}\mathcal{L}_{matter}}{\partial g^{\mu\nu}}$$

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Irrelevant $T\bar{T}$ deformation

Energy-Momentum Tensor: $T_M^{\mu\nu} = g^{\mu\nu}\mathcal{L}_M - \frac{\partial \mathcal{L}_M}{\partial(\partial_\mu A_\lambda)}(\partial^\nu A_\lambda)$

Maxwell theory: $T_M^{\mu\nu} = F^{\mu\lambda}F^\nu_\lambda - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}$

By choosing the deformation operator as: $O_\lambda = \frac{1}{8}\Big(T_{\mu\nu}T^{\mu\nu} - \frac{1}{2}T_\mu{}^\mu T_\nu{}^\nu\Big)$

Using the perturbation approach the extra terms add to Maxwell theory:

$$
\begin{aligned}
\mathcal{L}_\lambda &= \mathcal{L}_{Max} + \int O_\lambda d\lambda \\
&= S + \tfrac{1}{2}\lambda(S^2 + P^2) + \tfrac{1}{2}S\lambda^2(S^2 + P^2) + \dots \\
&= \frac{1}{\lambda}\left[1 - \sqrt{1 - \lambda\Big(2S + \lambda P^2\Big)}\right] = \mathcal{L}_{BI}
\end{aligned}
$$

Flow equation of Born-Infeld theory: $$\frac{\partial \mathcal{L}_{BI}}{\partial \lambda} = \frac{1}{8}\Big(T_{\mu\nu}T^{\mu\nu} - \frac{1}{2}T_\mu{}^\mu T_\nu{}^\nu\Big)$$

R. Conti, L. Iannella, S. Negro and R. Tateo, JHEP **11**, 007 (2018),

Generalized Born-Infeld theory by: $S \implies \mathcal{L}_{MM}$

$$\mathcal{L}_{GBI} = \frac{1}{\lambda}\left[1 - \sqrt{1 - \lambda\Big(2\mathcal{L}_{MM} + \lambda P^2\Big)}\right], \quad \mathcal{L}_{MM} = \cosh(\gamma) S + \sinh(\gamma)\sqrt{S^2 + P^2}.$$

$T\bar{T}$ **deformations in GBI theory:**

Irrelevant flow equation of GBI:

$$\frac{\partial \mathcal{L}_{GBI}}{\partial \lambda} = \frac{1}{8}\Big(T_{\mu\nu}T^{\mu\nu} - \frac{1}{2}T_{\mu}{}^{\mu}T_{\nu}{}^{\nu}\Big)$$

Root flow equation of GBI:

$$\frac{\partial \mathcal{L}_{GBI}}{\partial \gamma} = \frac{1}{2}\sqrt{T_{\mu\nu}T^{\mu\nu} - \frac{1}{4}T_{\mu}{}^{\mu}T_{\nu}{}^{\nu}} = \mathcal{R}_{\gamma}$$

- H. B.-A., Komeil Babaei Velni, Davood Mahdavian Yekta and H. Mohammadzadeh, "Emergence of non-linear electrodynamic theories from $T\bar{T}$-like deformations", Phys. Lett. B 829 (2022) 137079.

Einstein-Hilbert action by matter sector:

$$S_\lambda[g,\Phi] = S^{\text{EH}}[g] + S^M_\lambda[g,\Phi]$$

Action details:

$$S^{\text{EH}}[g] = \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^d\mathbf{x}\sqrt{g}\, R\,, \quad \frac{\partial\, S^M_\lambda[g,\Phi]}{\partial\,\lambda} = \int_{\mathcal{M}} \mathrm{d}^d\mathbf{x}\sqrt{g}\,\mathcal{O}_\lambda$$

Irrelevant $T\bar{T}$ operator

$$\mathcal{O}_\lambda = \frac{1}{2}\left(T_{\lambda,\mu\nu}T^{\mu\nu}_\lambda - r(T^{\mu}_{\lambda,\mu})^2\right)$$

Metric flow equation by Tateo Approach:

$$\frac{\partial g_{\mu\nu}}{\partial\lambda} = -2\frac{\partial\mathcal{O}_\lambda}{\partial T^{\mu\nu}_\lambda} = -2(T_{\lambda,\mu\nu} - rT^{\rho}_{\lambda,\rho}g_{\mu\nu}),$$

T. Morone and R. Tateo, [arXiv:2411.10265].

Solutions to the Ricci Flow via Einstein Field Equations

Analogous Ricci-Bourguignon flow equation:

$$\frac{d\,\tilde{g}_{\mu\nu}}{d\,\lambda} = -2(\,\tilde{R}_{\mu\nu} - (r - \frac{d\,r}{2} + \frac{1}{2})\,\tilde{R}\,\tilde{g}_{\mu\nu})$$

- In the case of $r = \frac{1}{(d-2)}$, we have a standard Ricci flow equation :

$$\frac{d\,\tilde{g}_{\mu\nu}}{d\,\lambda} = -2\,\tilde{R}_{\mu\nu}$$

- In the case of $r = 0$, we have a standard Einstein Ricci flow equation :

$$\frac{d\,\tilde{g}_{\mu\nu}}{d\,\lambda} = -2\,\tilde{G}_{\mu\nu}$$

- In the case of $r = \frac{1}{d}$, we have a standard traceless Ricci flow equation :

$$\frac{d\,\tilde{g}_{\mu\nu}}{d\,\lambda} = -2(\,\tilde{R}_{\mu\nu} - \frac{1}{d}\,\tilde{R}\,\tilde{g}_{\mu\nu})$$

- In the case of $r = \frac{1}{(d-1)}$, we have a standard Schouten Ricci flow equation:

$$\frac{d\,\tilde{g}_{\mu\nu}}{d\,\lambda} = -2(\,\tilde{R}_{\mu\nu} - \frac{1}{2(d-1)}\,\tilde{R}\,\tilde{g}_{\mu\nu})$$

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Root-Ricci Flow Equations

$$\mathcal{S}(g_{\mu\nu})_{\lambda,\gamma} = \mathcal{S}(g_{\mu\nu})_{\lambda,\gamma} + \delta\lambda \int d^dX\sqrt{g}\,\mathcal{O}(g_{\mu\nu})_{\lambda} + \delta\gamma \int d^dX\sqrt{g}\,\mathcal{O}(g_{\mu\nu})_{\gamma}$$

Let $\tau = \frac{\delta\gamma}{\delta\lambda}$.

Taylor expand:

$$\hat{\mathcal{S}}(h_{\mu\nu}) = \mathcal{S}_{\lambda,\gamma}(g_{0\mu\nu}) \quad + \quad \int d^dX\sqrt{g}\Big[c\left(h_{\alpha\beta}h^{\alpha\beta} - \frac{r}{dr-1}h_{\alpha}{}^{\alpha}h^{\beta}{}_{\beta}\right) - \tfrac{1}{2}h^{\alpha\beta}T_{\alpha\beta}$$
$$+ \quad \tau\sqrt{h_{\alpha\beta}h^{\alpha\beta} - \frac{1}{d}h_{\alpha}{}^{\alpha}h^{\beta}{}_{\beta}}\Big]\delta\lambda$$

$$\mathcal{T}_{\mu\nu} = \tau\left(\frac{\tilde{h}_{\mu\nu} - \frac{1}{d}\tilde{h}^{\alpha}{}_{\alpha}g_{\mu\nu}}{\sqrt{\tilde{h}_{\alpha\beta}\tilde{h}^{\alpha\beta} - \frac{1}{d}\tilde{h}_{\alpha}{}^{\alpha}\tilde{h}^{\beta}{}_{\beta}}}\right) + 2\,c\left(\tilde{h}_{\mu\nu} - \frac{r}{dr-1}\tilde{h}^{\alpha}{}_{\alpha}g_{\mu\nu}\right) - \frac{1}{2}T_{\mu\nu} = 0$$

Root-Ricci Flow Equations

Saddle point :

We find:

$$\left(\tau+2\,c\sqrt{\tilde{h}_{\alpha\beta}\tilde{h}^{\alpha\beta}-\frac{1}{d}\tilde{h}_{\alpha}{}^{\alpha}\tilde{h}^{\beta}{}_{\beta}}\right)^2-\frac{1}{4}\left(T_{\beta\mu}T^{\beta\mu}-\frac{1}{d}T^{\beta}{}_{\beta}T^{\mu}{}_{\mu}\right)=0$$

Can be written the saddle point as:

$$\tilde{h}_{\mu\nu}=\frac{\tau}{2\,c}\frac{(T_{\mu\nu}-\frac{1}{d}T^{\alpha}{}_{\alpha}g_{\mu\nu})}{\sqrt{T_{\beta\alpha}T^{\beta\alpha}-\frac{1}{d}T^{\beta}{}_{\beta}T^{\alpha}{}_{\alpha}}}+\frac{1}{4\,c}(T_{\mu\nu}-rT^{\alpha}{}_{\alpha}g_{\mu\nu})$$

By Tateo approach in matter sector:

$$\frac{d\,\tilde{g}_{\mu\nu}}{d\,\lambda}=-2\left[(1+\frac{2\tau}{\sqrt{\tilde{R}_{\alpha\beta}\tilde{R}^{\alpha\beta}-\frac{1}{d}\tilde{R}^2}})R_{\mu\nu}-(r-\frac{d\,r}{2}+\frac{1}{2}-\frac{2\tau}{d\sqrt{\tilde{R}_{\alpha\beta}\tilde{R}^{\alpha\beta}-\frac{1}{d}\tilde{R}^2}})Rg_{\mu\nu}\right]$$

Root-Ricci Flow Equations

- $T^{\alpha}{}_{\alpha} = 0$

$$\tilde{h}_{\mu\nu} = -4\tau \frac{T_{\mu\nu}}{\sqrt{T_{\alpha\beta}T^{\alpha\beta}}} - 2T_{\mu\nu}$$

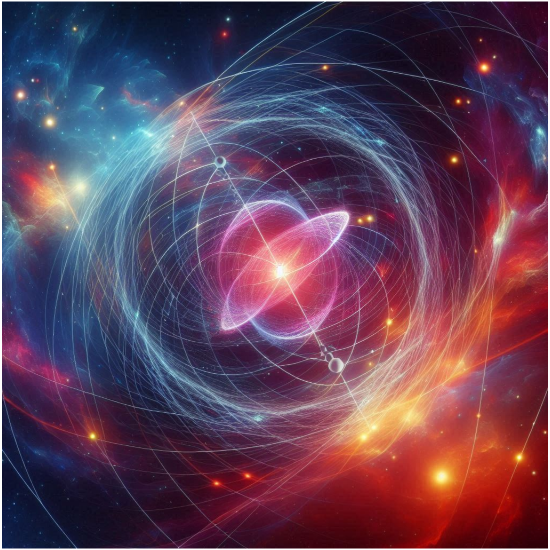
$$\frac{d\,\tilde{g}_{\mu\nu}}{d\,\lambda} = -2\Big(1 + \frac{2\tau}{\sqrt{4R_{\alpha\beta}R^{\alpha\beta} + (-4+d)R^2}}\Big)G_{\mu\nu}$$

In Limit $d = 4$:

$$\frac{d\,\tilde{g}_{\mu\nu}}{d\,\lambda} = -2\Big(1 + \frac{2\tau}{\sqrt{R_{\alpha\beta}R^{\alpha\beta}}}\Big)G_{\mu\nu}$$

- In Limit $r = \frac{1}{d}$:

$$\frac{d\,\tilde{g}_{\mu\nu}}{d\,\lambda} = -2\Big(1 + \frac{2\tau}{\sqrt{\tilde{R}_{\alpha\beta}\tilde{R}^{\alpha\beta} - \frac{1}{d}\tilde{R}^2}}\Big)(\,\tilde{R}_{\mu\nu} - \frac{1}{d}\,\tilde{R}\,\tilde{g}_{\mu\nu})$$



Thank you for your a$T\bar{T}$ention!