Generalized Liouville Action and Orbifold Riemann Surfaces

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- Recently, the Orbifold Riemann surfaces (Riemann surfaces with conical singularities and/or punctures) have seen renewed interest from different points of view (I,II,III,IV,V,VI,...) in (Quantum)Gravity.

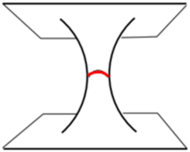
(*I*) :

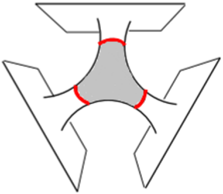
- The entanglement structure of quantum theory seems has a prominent role in the emergence of classical space-time:
  - The eternal BH with two asymptotic boundaries connected by ER bridge $\equiv$ TFD state that is an entangled state on two boundaries [Maldacena,2001].
  - Ryu-Takayanagi (RT) formula relates the amount of entanglement between the spatial sub-region with its complement in QFT state to area of minimal surface in AdS anchored on the boundary of sub-region [Ryu-Takayanagi,2006].
  - The (c)MERA ansatz for calculating ground state wavefunctions gives the additional dimension for AdS spacetime [Swingle,2009].
  - The linearized Einstein equations can be derived from the entanglement of the underlying quantum degrees of freedom [Lashkari, McDermott, Van Raamsdonk,2013].

Bi-partite entanglement

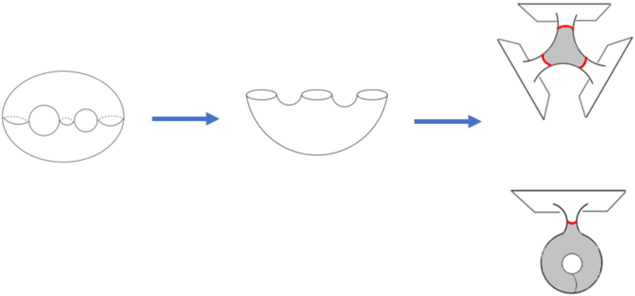
- But, the degrees of freedom can be entangled in multi-partite form. This is similar to this fact that the many-point correlation functions can not be inferred from the lower correlations. For example two states with tripartite entanglement are

$$|GHZ\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}, \quad |W\rangle = \frac{|001\rangle + |010\rangle + |100\rangle}{\sqrt{3}}$$





- These states can be defined by integrating over half of a higher genus surface .
- For one of the cutting the surface in half, the obtained slice has several components $\equiv$ the state lives in three copies of a (Hilbert space of) CFT. Actually, tripartite entanglement is the source of the bulk connectedness.
- For another one, the entanglement changes the global topology and creates non-trivial genus behind the horizon.



II:

- Another motivation comes from the consideration of Rényi entropies. When ρ is the reduces density matrix of some spatial region A,

$$S_n = \frac{1}{1-n} \log \mathrm{Tr}\,(\rho^n) = \frac{1}{1-n} (\log Z_n - n \log Z_1)$$

- It is generalization of von Neumann entropy (entanglement entropy) $S = -\mathrm{Tr}\,(\rho \log \rho)$ that can be obtained as the limit $n \to 1$.
- The Rényi entropies for integer "n", can be calculated by a path integral on a replicated surface with singular metric. This surface which is formed by joining together "n"-copies of original system across the region A, has genus $g = (N-1)(n-1)$ (for N disjoint entangling regions) which lives in a subspace of the moduli space with a $\mathbb{Z}_n$ symmetry.

III:

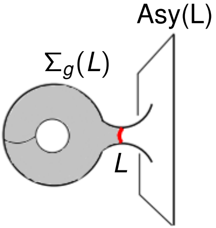
- In almost QM descriptions of BHs the geometric structure of the (individual) quantum microstates is unclear:
  - In String theory, D-Brane (BPS) bound states describes the microstates of special SUSY BH (five-dimensional extremal BH). One can count the number of these states but one has not control at finite coupling [Strominger and Vafa,1996].
  - In the AdS/CFT correspondence, the dual quantum states of BHs are known, but one has not clear description for the interior of BHs [Maldacena,1997].
  - In Cardy (like) approach, the symmetry analysis (near conformal boundary and(or) near Horizon geometry), gives the symmetry group algebra which by knowing its representation one can count the number of quantum states. But again no clear description for the interior of BHs.
- Knowing the geometric structure of microstates not only allow us to understand the origin of BH entropy but also the emergence of the classical spacetime from coarse-graining of microstates.[1]

[1] The fuzzball (no event horizon) proposal [Mathur,2005] is an example.

- Let's study the microstates of BH in 3D (EH theory). It seems impossible in 3D since the theory does not contain any local degrees of freedom. But do not forget the global degrees of freedom associated with the topology of spacetimes that can be quantized.

- In the FRW coordinate $ds^2 = -dt^2 + cos^2(t)d\Sigma^2$, the microstate geometries are defined by hyperbolic metric on a smooth surface Σ_g of genus $g \geq 1$. This surface has one hole ($\partial\Sigma_g = S_1$) and the geometry of asymptotic boundary intersects with Σ_g along S_1.

$$\Sigma_g = \Sigma_g(L) \cup \text{Asy(L)}$$



- The surface Asy(L) is an annulus, with one infinite length boundary matching onto the conformal boundary of AdS_3 and one geodesic boundary with length L. These information fix uniquely the geometry of this surface: the moduli space (different configuration space) seems trivial. But wait...
- To define a theory on the asymptotically AdS_3 spacetimes, one must demand the proper boundary conditions (Brown and Henneaux) on the asymptotic metric. The diffeomorphism ($diff(S^1)$) which act non-trivially on the conformal boundary generates symmetries which their conserved charges (after proper quantization) give two copies of Virasoro algebra.
- The states that are obtained from this quantization are known as boundary graviton since their appearance comes from presence of a boundary ($\sim$ edge states in quantum hall systems).
- Therefore, the quantization of $\Sigma_g(L)$ leads to a BH microstate ($\equiv$ primary state in the CFT_2) which is dressed by boundary gravitons (descendant states).

- Quantization of $\Sigma_g(L)$:
  - A pair of pants is a sphere with three boundary geodesics (holes or cuffs) that they determine uniquely the constant negative curvature metric on the pair of pants.
  - Sewn the pair of pants together along their geodesic boundaries.
  - Accordingly, the metric on $\Sigma_g(L)$ is determined by $(3g-3)+1$ complex parameters (internal geodesics length L_i + internal twists τ_i + length L).[2] We call this space as the moduli space $\mathcal{M}_{g,1}(L)$ ($\equiv$ space of classical solutions = Phase space).[3]
  - Its symplectic structure is ($k = 1/16G_N$)

$$\omega = \frac{1}{2\pi^2}\left(LdL \wedge d\tau + \sum_j L_j dL_j \wedge d\tau_j\right) = \frac{1}{4\pi}\omega_{WP}(\mathcal{M}_g(L))$$

[2] L_i, τ_i are known as Fenchel-Nielsen coordinates on moduli space. There is also a twist parameter τ which describes how the black hole interior is matched onto asymptotic infinity.

[3] To be more precise, the parameters (L_i, τ_i) parameterize the Teichmüller space $T_{g,1}(L)$ of bordered Riemann surfaces and (for chiral gravity)

$$\mathcal{M}_{g,1}(L) = T_{g,1,}(L)/MCG(\Sigma_g(L)).$$

- By knowing the symplectic structure, one can quantize the phase space by promoting the moduli space coordinates to operators acting on the Hilbert space. For example, for the area L of the event horizon ("Geomtric Quantization")

$$[L^2,\tau]=i\frac{4\pi^2}{k}\quad\to\quad E=\frac{1}{4\pi^2}kL^2\in\mathbb{Z}.$$

- At the semi-classical approximation, the number of genus g microstates with energy E is just the volume of $\mathcal{M}_{g,1}(L)$. For ψ_1 the curvature of line bundle over $\mathcal{M}_{g,1}$ associated to cotangent bundle at puncture and κ_1 the first tautological class $[\omega_{WP}]=2\pi^2\kappa_1$, [Maloney,2015]

$$N_g(k,E)=\frac{1}{(3g-2)!}\int_{\mathcal{M}_{g,1}}\left(\frac{k}{2\pi^2}\omega_{WP}(\mathcal{M}_g(L))\right)^{3g-2}$$

$$\overset{[Mirzakhani,2007]}{=}\frac{1}{(3g-2)!}\int_{\mathcal{M}_{g,1}}\left(\frac{k}{2\pi^2}\omega_{WP}(\mathcal{M}_{g,1})+E\psi_1\right)^{3g-2},$$

$$\overset{[Witten,1991-Kontsevich,1992-Mirzakhani,2011]}{\sim}\sum_{a=0}^{3g-2}c(g,E,a)\underbrace{\int_{\mathcal{M}_{g,1}}\kappa_1^{3g-2-a}\psi_1^a}_{\text{intersection numbers on moduli space}}\sim E^{3g-2}\ll e^{4\pi\sqrt{kE}}$$

- Apart the above mentioned problem, if one add the contribution of all genus, the result will diverges. Therefore, to obtain the correct number of microstates:
  - One might to study the full quantum Hilbert space via geometric quantization.
  - It might be distinct geometries with different genus are related very secretly together by some new type of gauge symmetry.
  - Pure gravity might not exist and one needs some extra matters such as massive particles ($\equiv$ conical singularities)

(IV) :

- If one uses only the smooth saddle-points in gravitational path integral of 3D gravity, the obtained partition function suffers from two problems:
  - The spectrum of twists at fixed spin is continuous [Maloney, Witten,2010] rather than discrete. This problem potentially can be resolved by noting to the very recent developments which state that the dual theory to the 2D AdS gravity is an ensemble of 1-dimensional QMs, [Saad,Shenker,Stanford,2019].
  - Existence of negative degeneracy of states for large spins J and energies E very close to the Edge of spectrum[4] [Benjamin, Ooguri, Shao, Wang, 2019]

$$\rho_J(E) \sim A_0 e^{S_0(J)} \sqrt{E-|J|} + A_1 (-1)^J e^{S_0(J)/2} \frac{1}{\sqrt{E-|J|}}$$

The first term, come from the ordinary BTZ and the second term which comes from the simplest class of $SL(2,\mathbb{Z})$ black holes, is exponentially suppressed by a relative factor of $e^{S_0(J)/2}$ but enhanced very close to the edge.

[4] $E = h + \bar{h} - \frac{c-1}{12}, J = \bar{h} - h$

The minimal cure for the non-unitarity problem is provided by two different groups and different ideas:

- [Benjamin,Collier, Maloney,2020]:
  One should include the additional states below the black hole threshold which their geometric interpretation is that they are conical defects with deficit angle $\delta\phi = 2\pi(1 - 1/m)$ where m is a positive integer.[5] The matter loop runs round the horizon of BTZ and gives extra contribution $e^{a_m S_0(J)}(\sqrt{E - |J|})^{-1/2}$, where for a scalar of (local) mass "$\mathfrak{m}$", $a_m = 1 - 4\mathfrak{m}G_N$. For $\mathfrak{m} \sim 1/8G_N$, the density grows fast enough to cure the negativity.

[5] The defects with deficit angle $\delta\Phi$ can be seen a massive particles with masses $\mathfrak{m} \sim \delta\phi/G_N$ in AdS_3.

- [Maxfield and Truiaci,2020] proposed an alternative resolution which does not require a modification of the theory, but rather the inclusion of additional contributions to the path integral over metrics.

- Actually, over spacetime topologies (Seifert manifolds) for which no classical solution exists($\equiv$ "off-shell" path integral)

$$\begin{aligned}\rho_J(E) &\sim A_0 e^{S_0(J)}\sqrt{E-|J|} + A_1(-1)^J e^{S_0(J)/2}\frac{1}{\sqrt{E-|J|}}\\ &+ A_2(E-|J|)^{-3/2} + A_3(-1)^J e^{-S_0(J)/2}(E-|J|^{-5/2}) + \ldots\\ &\sim A_0 e^{S_0(J)}\sqrt{E-E_0(J)}\end{aligned}$$

where $E_0(J) - |J| \sim -(-1)^J e^{-S_0(J)/2}$. This means that the non-unitary problem is an artifact of truncating the binomial series.

- The 2D compactified theory contains nonperturbative conical defects (JT gravity with defects [Witten,2020]).

Schottky Uniformization:

- A marked Schottky group Σ of rank g is a relation-free system of generators $L_1, \ldots, L_g \in PSL(2, \mathbb{C})$ that act properly on Ω.
- If the fixed-points of Σ are encoded in limit set Λ then the region of discontinuity is $\Omega = \hat{\mathbb{C}} \backslash \Lambda$.
- For every marked Schottky group there is a *fundamental domain* $\mathcal{F}$ for Σ in Ω which is a (connected) region in $\hat{\mathbb{C}}$ bounded by $2g$ disjoint Jordan curves $C_1, \ldots, C_g, C'_1, \ldots, C'_g$ with $C'_i = -L_i(C_i)$, $i = 1, \ldots, g$. Each element L_i can be represented in the *normal form*

$$\frac{L_i(w) - a_i}{L_i(w) - b_i} = \lambda_i \frac{w - a_i}{w - b_i}, \qquad w \in \hat{\mathbb{C}},$$

where a_i and b_i and $0 < |\lambda_i| < 1$ are the attracting, repelling fixed points of the loxodromic element L_i and multiplier, respectively.

- The Schottky groups can be used to construct genus "g" surfaces, Ω/Σ.

- In the saddle-point approximation, the partition function on that higher genus surface is related to the classical Liouville action [Zograf and Takhtajan,1988 (was later interpreted by [Takhtajan and Teo,2002] in cohomological language):

$$S[\varphi] = \iint_{\mathcal{D}} (|\partial_w \varphi|^2 + e^{\varphi}) d^2 w + \frac{\sqrt{-1}}{2} \sum_{k=2}^{g} \oint_{C_k} \theta_{L_k^{-1}}(\varphi),$$

where the 1-form $\theta_{L_k^{-1}}(\varphi)$ is given by $(\forall\, L_k \in (G; L_1, \ldots, L_g))$

$$\theta_{L_k^{-1}}(\varphi) = \left(\varphi - \frac{1}{2}\log|L_k'|^2 - \log|l_k|^2\right)\left(\frac{L_k''}{L_k'}dw - \frac{\overline{L_k''}}{\overline{L_k'}}d\bar{w}\right),$$

and $l_k = \frac{1-\lambda_k}{\sqrt{\lambda_k}(a_k-b_k)}$ is the left-hand lower element in the matrix representation of the generator $L_k \in \mathrm{PSL}(2,\mathbb{C})$ for $k = 2, \ldots, g$.

V:

- The above Schottky uniformisation picture can be extended to a quotient of $\mathbb{H}_3$. The results of that are handlebody solution in a Euclidean bulk, holographically dual to the field theory on the Riemann surface.

- Infinitely many Schottky uniformisations, different handlebodies, characterised by the choice of boundary cycles which become contractible in the bulk. The stable solution is determined by the least action principle and it dominates in the calculation of the partition function and therefore determines wavefunction of gravitational system.

- It plays an important role in the proof of Ryu-Takayanagi formula and specially whether the the assumption of replica symmetry is true or not in the dual gravitational system.[6]

[6] A preliminary results are provided by [Lewkowycz,Maldacena,2013] [Faulkner, 2013].

(VI):

- Apart from that important problems in Physics, studying orbifold Riemann surfaces (+ punctures) might help to prove some important conjectures or new theorems in Mathematics. Let us remind couple of them.

- [Klein,1883] and [Poincaré,1884], consider the following second-order linear differential equation on a Riemann surface X with genus 0 and n-punctures ($m \to \infty$)

$$\frac{d^2 Y}{dw^2} + \frac{1}{2} Q_X(w) Y = 0, \quad w \in X$$

where

$$Q_X(w) = \sum_{i=1}^{n-1} \left(\frac{1}{2(w - w_i)^2} + \frac{c_i}{w - w_i} \right).$$

- For given c_is (saccessory parameters), they could found analytic continuation of the ratio Y_1/Y_2 of two linearly independent solutions Y_1 and Y_2 along all non-contractible loops in X which leads to a monodromy representation Mon : $\pi_1(X) \to PSL(2, \mathbb{C})$. Its image in $PSL(2, \mathbb{C})$ is called the monodromy group Γ (with parabolic generators $P_1 ... P_n = 1$). They called it "uniformization theorem":

$$X \cong \mathbb{H}/\Gamma \qquad Y_1/Y_2 = J^{-1} : X \to \mathbb{H}$$

- Since the time of Klein and Poincare, c_is were considered to be "mysterious" object. Not only one could not compute them explicitly but also prove that they exist and are single-valued functions of the punctures.
- But Physics (Quantum Liouville Theorem and its Conformal Ward Identities) can give some important information about them even the ones that Klein and Poincare were not aware of them!

- It is well-known that the response of a CFT to the Weyl transformation is encoded in classical Liouville action.
- Accordingly, the quantized Liouville theorem can describe the quantum corrections to those (hyperbolic) geometries.
- This provide one way to 2D quantum gravity via the quantum Liouville theorem by demanding that quantum Liouville theorem takes the advantage of conformal symmetry as its classical cousin.
- If the conformal symmetry is also the symmetry of quantum Liouville theorem, it must show itself in the conformal Ward identities (CWIs) for correlation functions of components of energy-momentum tensor with another operators.

- For $X_{\boldsymbol{m}} = V_{m_1}(w_1)\cdots V_{m_n}(w_n)$ that V_{m_i} are some vertex operators $V_{m_i} = e^{\alpha(m_i)\psi/\hbar}$ $(m_i \to \infty)$, the quantum Liouville theorem is defined by

$$\langle X_{\boldsymbol{m}} \rangle = \int_{CM_{\boldsymbol{m}}(X)} \mathcal{D}\psi \; e^{-\frac{1}{2\pi\hbar} S_{\boldsymbol{m}}[\psi]},$$

  where the $S_{\boldsymbol{m}}$ is the quantum Liouville action in presence of some punctures. Moreover,

$$\langle \boldsymbol{T}(w) X_{\boldsymbol{m}} \rangle = \int_{CM_{\boldsymbol{m}}(X)} \mathcal{D}\psi \; \boldsymbol{T}(\psi)(w) \; e^{-\frac{1}{2\pi\hbar} S_{\boldsymbol{m}}[\psi]}.$$

- To have conformal symmetry, one actually requires to prove that

$$\frac{1}{\hbar}\langle \boldsymbol{T}(w) X_{\boldsymbol{m}} \rangle = \sum_{i=1}^{n} \left(\frac{\boldsymbol{h}_{m_i}(\hbar)}{(w-w_i)^2} + \frac{1}{(w-w_i)}\partial_{w_i} \right) \langle X_{\boldsymbol{m}} \rangle, \tag{1}$$

  with $\boldsymbol{h}_{m_i}(\hbar)$ are considered as conformal dimensions of those vertex operators.

Note that at the tree level when $\hbar \to 0$,

$$\langle X_{\boldsymbol{m}} \rangle \sim e^{-\frac{1}{2\pi\hbar} S_{\boldsymbol{m}}[\varphi]}, \qquad \boldsymbol{h}_{m_i}(\hbar) \sim \frac{h_{\rm cl}(m_i)}{2\hbar}$$
$$\langle \boldsymbol{T}(w) X_{\boldsymbol{m}} \rangle \sim T_{\varphi}(w)\, e^{-\frac{1}{2\pi\hbar} S_{\boldsymbol{m}}[\varphi]}.$$

By substituting the above relations in (1), one gets

$$\frac{1}{\hbar}\, T_{\varphi}(w)\, e^{-\frac{1}{2\pi\hbar} S_{\boldsymbol{m}}[\varphi]} = \sum_{i=1}^{n} \left(\frac{h_{\rm cl}(m_i)}{2\hbar (w-w_i)^2} + \frac{1}{(w-w_i)} \partial_{w_i} \right) e^{-\frac{1}{2\pi\hbar} S_{\boldsymbol{m}}[\varphi]},$$

which implies that

$$T_{\varphi}(w) \; = \sum_{i=1}^{n} \left(\frac{h_{\rm cl}(m_i)}{2(w-w_i)^2} - \frac{1}{2\pi} \frac{1}{(w-w_i)} \partial_{w_i} S_{\boldsymbol{m}}[\varphi] \right).$$

Using the expression of EMT of classical Liouville action, $T_{\varphi}(w)$, gives

$$h_{\rm cl}(m_i) = 1 - \frac{1}{m_i^2}\, (= 1, \text{puncture}), \qquad \partial_{w_i} S_{\boldsymbol{m}}[\varphi] = -2\pi c_i,$$

The classical Liouville action is the generating function of the c_is! Very interestingly, it was conjectured by [Polyakov,1982]!!

- Moreover, the conformal symmetry at the quantum level implies that the vertex operators and components of the energy-momentum tensor satisfy the OPE of BPZ[7], for example

$$\frac{1}{\hbar^2}\boldsymbol{T}(w)\overline{\boldsymbol{T}}(\bar{w}') = \ldots \quad \text{(regular terms)}$$

which yields the following CWIs,

$$\frac{1}{\hbar^2}\langle \boldsymbol{T}(w)\overline{\boldsymbol{T}}(\bar{w}')X_{\boldsymbol{m}}\rangle = \frac{1}{\hbar}\sum_{i=1}^{n}\left(\frac{\boldsymbol{h}_{m_i}(\hbar)}{(w-w_i)^2}+\frac{1}{(w-w_i)}\partial_{w_i}\right)\langle \overline{\boldsymbol{T}}(\bar{w}')X_{\boldsymbol{m}}\rangle.$$

- For the $\langle \boldsymbol{T}(w)\overline{\boldsymbol{T}}(\bar{w}')X_{\boldsymbol{m}}\rangle_{\mathrm{nc}}$, the similar tree level analysis $\hbar \to 0$, gives [Takhtajan,1994]

$$\frac{\partial^2 S_{\boldsymbol{m}}[\varphi]}{\partial w_i \partial w_j} = (\text{WP metric})_{ij} \qquad \rightarrow \qquad \frac{\partial c_i}{\partial w_j} = \frac{\partial c_j}{\partial w_i}$$

where Neither Klein nor Poincare were aware of these results! and [Polyakov and Zamolodchikov,1982-1984] just had a conjecture for it!!

[7] Belavin-Polyakon-Zamolodchikov

- In collaboration with Kuroush Allameh (USC), Behrad Taghavi (IPM) and by using the modern language of uniformization we studied the orbifold Riemann surfaces with non-trivial genus and found the proper generalized Liouville action (2310.17536):

$$\mathbb{S}_{\boldsymbol{m}} = S_{\boldsymbol{m}} - \pi \sum_{i=1}^{n} (m_i - \tfrac{1}{m_i}) \log h_i,$$

$$h_i = \begin{cases} \left| J_1^{(i)} \right|^{\frac{2}{m_i}} & i = 1, \ldots, n_e, \\ \left| J_1^{(i)} \right|^2 & i = n_e + 1, \ldots, n-1, \\ \left| J_{-1}^{(n)} \right|^2 & i = n. \end{cases}$$

- Intuition: Generalized Liouville action $\mathbb{S}_{\boldsymbol{m}}$ should be independent from the choice of a fundamental domain.

- The modern quantum geometry of strings primarily explores all surfaces, analyzing variations in their metrics and the determinants of the corresponding Laplacians.
- A key area of interest is how this determinant, viewed as a function of the metric on a given compact surface, behaves, particularly in identifying its extreme values under specific metric constraints.
- This topic is extensively examined in the seminal work by Osgood, Phillips, and Sarnak [1988]. They investigated the function $-\log\det\Delta$ as a height function on the space of metrics for a compact, orientable, smooth surface of genus g.
- They discovered that for surfaces with $g > 1$, this function reaches its minimum at the unique (up to scaling, and treating isometric surfaces as equivalent) hyperbolic metric within any given conformal class of metrics and has no other critical points.

- Their result can be viewed as new perspective on the classical uniformization theorem:
- The uniformization theorem states that every simply connected Riemann surface can be conformally mapped to one of three canonical geometries: the sphere (positive curvature), the Euclidean plane (zero curvature), or the hyperbolic plane (negative curvature).
- For surfaces of genus greater than 1, the uniformization theorem implies that the surface admits a unique hyperbolic metric within its conformal class.

- On the other hand, the Liouville action offers a variational principle where the critical point correspond to constant curvature metrics (such as hyperbolic metrics for surfaces with genus $g > 1$ which appears also from uniformization theorem).
- All of this indicates a profound connection between the spectrum of $-\log\det\Delta$ (and thus the Polyakov anomaly) and the Liouville action, identified by Takhtajan and Teo [2003] for compact Riemann surfaces.
- In collaboration with Hossein Mohammadi (SUT) and Behrad Taghavi (IPM), we demonstrate that this relationship can be extended to orbifold Riemann surfaces via our generalized Liouville action $\mathbb{S}_{\boldsymbol{m}}$.

- According to Kalvin [2019] or Aldana, Kirsten and Rowlett [2020], for the metric $\tilde{g} = e^{\sigma} g$, the Polyakov anomaly formula is given by

$$\log \frac{\det(\Delta_{\tilde{g}})}{A_{\tilde{g}}} - \log \frac{\det(\Delta_g)}{A_g}$$
$$= -\frac{1}{12\pi} \left(\iint_{D_\delta} \Big(\partial_z \sigma \partial_{\bar{z}} \sigma - K[\varphi] e^{\varphi} \sigma \Big) d^2 z + \pi \sum_{j=1}^{n_e} m_j h_j \sigma(z_j) \right) .$$

- The changes of our action (without area term) under conformal transformation

$$\mathbb{S}_{\boldsymbol{m}}[\varphi+\sigma] - \mathbb{S}_{\boldsymbol{m}}[\varphi] = \iint_{D_\delta} \Big(\partial_z \sigma \partial_{\bar{z}} \sigma + K[\varphi]\, e^{\varphi} \sigma \Big) d^2 z + \pi \sum_{j=1}^{n_e} m_j h_j\, \sigma(z_j).$$

- The method we employed to find this extension offers an alternative approach for deriving the renormalized Polyakov anomaly for Riemann surfaces with punctures (cusps).
- Assuming that relation remains valid even in the presence of punctures, we find that[8]

$$\Big(P[\varphi+\sigma]-P[\varphi]\Big)_{\text{ren}}=\Big(P[\varphi+\sigma]-P[\varphi]\Big)-\frac{1}{6}\lim_{\epsilon\to 0}\sum_{j=n_e+1}^{n}\Big(1-e^{-\sigma(z_j)/2}\Big)\log\epsilon$$

which leads to:

$$\Big(P[\varphi+\sigma]-P[\varphi]\Big)_{\text{ren}}=-\frac{1}{12\pi}\left(\iint_{\overset{\curlywedge}{\mathcal{D}}_\delta}\Big(\partial_z\sigma\partial_{\bar z}\sigma-K[\varphi]e^{\varphi}\sigma\Big)d^2z\right).$$

- Unlike conical singularities, where additional finite terms appear in the Polyakov anomaly compared to the compact case, punctures (cusps) do not introduce any new finite terms, Albin, Aldana, and Rochon [2013].

[8] Where $P[\varphi]=\log\frac{\det(\Delta_g)}{A_g}$ and $\epsilon=|z-z_j|$.

- Let define the bulk renormalized volume as follows

$$V_{\text{ren}} = \lim_{\varepsilon\to 0}\Bigg(V_\varepsilon[\varphi] - \frac{1}{2}A_\varepsilon[\varphi] - \pi\chi(X)\log\varepsilon \\ - \frac{\pi}{2}\sum_{j=1}^{n_e}(1-\frac{1}{m_j})^2\log\varepsilon - \frac{\pi}{2}n_p(\log\varepsilon + 2\log|\log\varepsilon|)\Bigg),$$

- Then, using the technology of double (co)homology complexes ((co)homology group and group (co)homology) we obtain

$$V_{\text{ren}} = -\frac{1}{4}\Big(\mathbb{S}_{\boldsymbol{m}}[\varphi] - \text{area term}\Big).$$

- The renormalized hyperbolic volume V_{ren} coincides with the generalized Liouville action $\mathbb{S}_{\boldsymbol{m}}$.
- The renormalized hyperbolic volume V_{ren} is inherently connected to the boundary's uniformization theory and follows a Polyakov-type anomaly formula.