Krylov Complexity as a Probe for Chaos

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Chaos

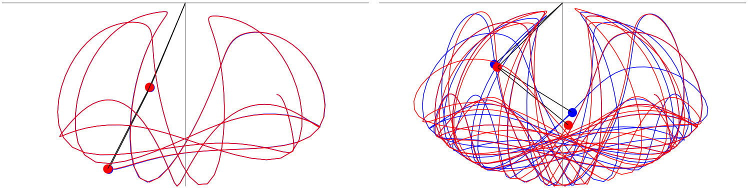
Classical Chaos

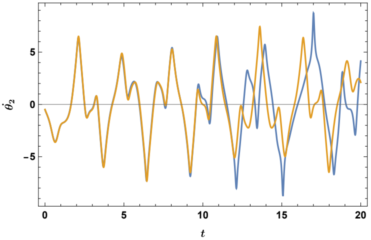
- In the Chaotic system trajectories in phase space show extreme sensitivity to initial conditions
- Nearby trajectories separate exponentially fast which Characterized by the Lyapunov exponent
- In a chaotic system, if the initial condition changes as $Y(0) \to Y(0) + \delta Y(0)$, the system's trajectory at any later time changes as $Y(t) \to Y(t) + \delta Y(t)$, such that

$$\left(\frac{\partial Y(t)}{\partial Y(0)} \right)^2 \sim e^{\lambda t}$$

  where λ is the Lyapunov exponent.
- For a integrable system with N degrees of freedom, there is N conserved quantities

Double pendulum





Quantum Chaos

- Applying concepts of classical chaos to quantum chaos involves several challenges
  - Quantum systems are governed by probabilistic laws and described by wavefunctions
  - Quantum mechanics does not have well-defined trajectories due to the uncertainty principle
  - Quantum coherence and superposition effects have no direct analogs in classical mechanics
- Two of the most widely used methods are out-of-time-order correlators (OTOCs) and spectral statistics
- For two observables $W(t)$ and $V(0)$ in the infinite temperature OTOC is defined

$$\mathrm{OTOC}(t) = -\frac{1}{d}\mathrm{Tr}([W(t), V(0)]^2)$$

- Exponential growth over time is recognized as a hallmark of quantum chaos

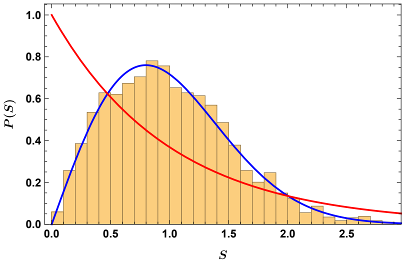
Level spacing

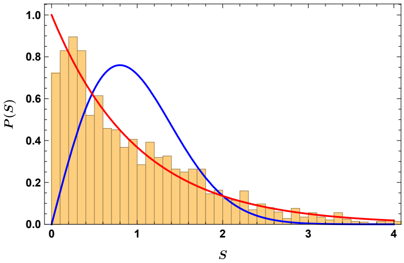
- Chaotic Hamiltonians exhibit the same energy level spacing as those predicted by Random Matrix Theory
- If E_n is eigenvalues of the Hamiltonian with the ordering $E_{n+1} > E_n$. The level spacing is defined by $S_n = E_{n+1} - E_n$
- $P(s)$ is probability density for two neighboring eigenenergies E_n and E_{n+1} having the spacing s

$$\begin{array}{ll} \text{Poisson} & P(s) = e^{-s}, \\ \text{Wigner} - \text{Dyson} & P(s) = As^{\delta} e^{-Bs^2} \quad \text{for } \delta = 1, 2, 4 \end{array}$$

- If the distribution is Poissonian, the model is integrable. For maximally chaotic systems, the distribution follows Wigner-Dyson statistics

Level spacing



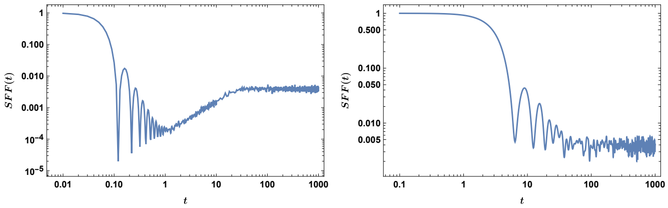


Spectral Form Factor (SFF)

- The SFF measures the correlation between eigenvalues at different time scales

$$\mathrm{SFF}(t) := \frac{|Z(\beta+it)|^2}{|Z(\beta)|^2} = \frac{1}{Z(\beta)^2}\sum_{m,n} e^{-\beta(E_m+E_n)}\, e^{i(E_m-E_n)t}$$

- The presence of a linear ramp in the SFF serve as signatures of chaos



Krylov basis

Krylov basis

- Time evolution of initial state

$$|\psi(t)\rangle = e^{iHt}|\psi_0\rangle \sum{}_n \frac{(-it)^n}{n!}|\psi_n\rangle \qquad |\psi_n\rangle = H^n|\psi_0\rangle$$

- Consider $|\psi_n\rangle$ as basis $\{H^0|\psi_0\rangle, H^1|\psi_0\rangle, H^2|\psi_0\rangle, ...\}$
- Recursively applies the Gram–Schmidt procedure to $|\psi_n\rangle$ to generate an orthonormal basis

Lanczos algorithm

1. The first element of the basis is identified with the initial state $|0\rangle \equiv |\psi_0\rangle$ and $b_0 = 0$
2. Other elements are constructed, recursively, as follows

$$|\widehat{n+1}\rangle = (H - a_n)|n\rangle - b_n|n-1\rangle \qquad |n\rangle = b_n^{-1}|\hat{n}\rangle$$

with Lanczos coefficients

$$a_n = \langle n|H|n\rangle \qquad b_n = \sqrt{\langle \hat{n}|\hat{n}\rangle}$$

Krylov basis

- This recursive procedure stops whenever b_n vanishes which occurs for $n = \mathcal{D}_\psi \leq \mathcal{D}$
- This new basis $\{|n\rangle, n = 0, 1, 2, \cdots, \mathcal{D}_\psi - 1\}$ is Krylov basis
- Hamiltonian be put into a tridiagonal form

$$H|n\rangle = a_n|n\rangle + b_{n+1}|n+1\rangle + b_n|n-1\rangle$$

- Time evolved state can be expanded in this basis

$$|\psi(t)\rangle = \sum_{n=0}^{\mathcal{D}_\psi - 1} \phi_n(t)\,|n\rangle\,, \quad \text{with} \quad \sum_{n=0}^{\mathcal{D}_\psi - 1} |\phi_n(t)|^2 = 1$$

- The wave function $\phi_n(t)$ satisfies the following equation

$$-i\partial_t\phi_n(t) = a_n\phi_n(t) + b_n\phi_{n-1}(t) + b_{n+1}\phi_{n+1}(t) \qquad \phi_n(0) = \delta_{n0}$$

Krylov Complexity

- Special operator in Krylov space whose matrix elements is exactly diagonal: Number Operator

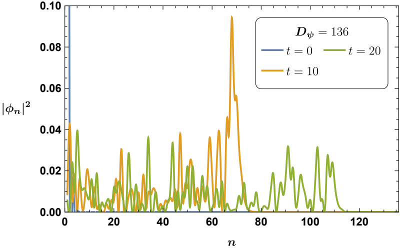
$$\mathcal{N} = \sum_{n=0}^{\mathcal{D}_\psi - 1} n|n\rangle\langle n|,$$

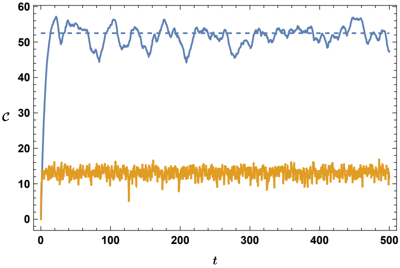
- The expectation value of this operator, computes Krylov complexity (Balasubramanian,2022)

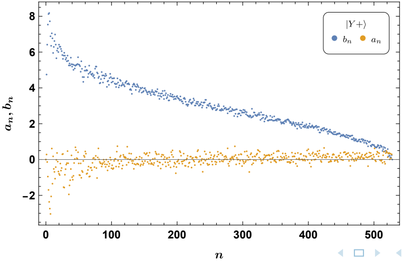
$$\mathcal{C}(t) = \langle \psi(t)|\mathcal{N}|\psi(t)\rangle = \sum_{n=0}^{\mathcal{D}_\psi - 1} n \langle n|\rho(t)|n\rangle = \sum_{n=0}^{\mathcal{D}_\psi - 1} n \, |\phi_n(t)|^2$$

- The infinite time average of Krylov complexity

$$\overline{\mathcal{C}} = \lim_{T\to} \frac{1}{T} \int_0^T \langle \mathcal{N}(t)\rangle \, dt = \sum_{n=0}^{\mathcal{D}_\psi - 1} n \; C_{nn}$$


$D_\psi = 136$
$t = 0$
$t = 20$
$t = 10$
$|\phi_n|^2$
n


$\mathcal{C}$
t


$|Y+\rangle$
b_n
a_n
a_n, b_n
n

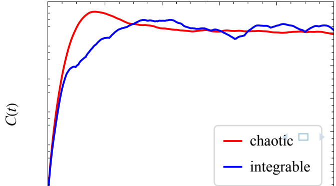
Chaos and Krylov Complexity

Chaos and Krylov Complexity

Why do we expect K-complexity can probe chaos?

K-complexity can measure chaos by evaluating how states spread within the Krylov subspace

- exponential growth of the K-complexity $K_O(t) \propto e^{K_t}$ could be interpreted as a signature of chaos (Parker,2018)
- K-complexity as a probe of the integrable or chaotic nature of the system via its late-time saturation value (Rabinovici,2022)
- Multiseed Krylov Complexity (Craps,2024)
- The exhibition of the peak prior to reaching its saturation value for maximally entangled state (Erdmenger,2023)



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What is the impact of different initial states on this observation?

Krylov Basis vs Energy Eigenbasis

- Consider a quantum system with a time-independent local non-degenerate Hamiltonian H with dimension $\mathcal{D}$

$$|E_\alpha\rangle : \text{Eigenstates} \quad E_\alpha : \text{Eigenvalues}$$

- The expansion of the initial state in terms of the energy eigenbasis

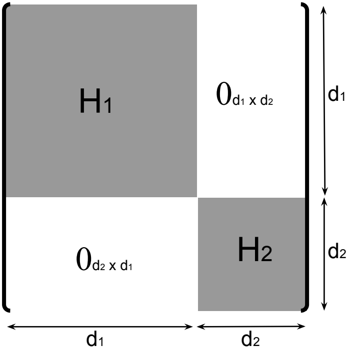
$$|\psi_0\rangle = \sum_{\alpha=1}^{\mathcal{D}} c_\alpha |E_\alpha\rangle$$

- The expansion of an element of the Krylov space in terms of the energy eigenbasis

$$|n\rangle = \sum_{\alpha=1}^{\mathcal{D}} f_{n\alpha} |E_\alpha\rangle \qquad \sum_{\alpha=1}^{\mathcal{D}} f^*_{n\alpha} f_{m\alpha} = \delta_{nm}$$

Reducing to Symmetry Block

- In general, the inequality $\mathcal{D}_\psi \leq \mathcal{D}$ stems from the symmetries in the model
- Eigenstates can be classified into distinct blocks based on their symmetry



Reducing to Symmetry Block

- In general, the inequality $\mathcal{D}_\psi \leq \mathcal{D}$ stems from the symmetries in the model
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- The dimension of the Krylov space matches that of the corresponding symmetry block
- By reducing to a symmetry block, the summation terminates at $\mathcal{D}_\psi$

$$|n\rangle = \sum_{j=1}^{\mathcal{D}_\psi} f_{nj} |E_j\rangle \,, \qquad \sum_{j=1}^{\mathcal{D}_\psi} f^*_{nj} f_{mj} = \delta_{nm}$$

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- In the symmetry block, f_{nj} will be invertible

$$|E_j\rangle = \sum_{j=1}^{\mathcal{D}_\psi} f_{nj} |n\rangle \qquad \sum_{n=1}^{\mathcal{D}_\psi} f^*_{nj} f_{nj} = \delta_{ij}$$

Number Operator In Energy Eigenbasis

- By writing the recursion relation in the energy eigenbasis, we can express f_{nj} in terms of $f_{0j} = c_j$

$$f_{nj} = \frac{g_n(E_j)}{b_n!} c_j$$

$$b_n! = b_1 \cdots b_n \qquad g_n(E_j) = \det(E_j - H^{(n)})$$

$$H^{(n)}_{pq} = \langle p|H|q\rangle = a_q \delta_{pq} + b_{q+1}\delta_{pq+1} + b_q \delta_{pq-1}\,, \quad p, q = 0, \cdots, n-1$$

- The explicit form of the matrix elements of the number operator in the energy eigenbasis

$$\mathcal{N}_{jk} = \sum_{n=0}^{\mathcal{D}_\psi - 1} n f_{nj} f^*_{nk} = c_j c^*_k \sum_{n=0}^{\mathcal{D}_\psi - 1} \frac{n g_n(E_j) g_n(E_k)}{(b_n!)^2}$$

Time Evolution of Krylov Complexity

- The time evolution of the Number operator in the energy eigenbasis

$$\mathcal{C}(t) = \langle \psi(t)|\mathcal{N}|\psi(t)\rangle = \sum_{j}^{\mathcal{D}_\psi} |c_j|^2 \mathcal{N}_{jj} + \sum_{j\neq k}^{\mathcal{D}_\psi} e^{i\omega_{jk}t} c_j^* c_k \mathcal{N}_{jk} \,,$$

- Using $\mathcal{C}(0) = 0$

$$\mathcal{C}(0) = \sum_{j,k=1}^{\mathcal{D}_\psi} c_j^* c_k \mathcal{N}_{jk} = 0 \longrightarrow \sum_{j}^{\mathcal{D}_\psi} |c_j|^2 \mathcal{N}_{jj} = -\sum_{j\neq k}^{\mathcal{D}_\psi} c_j^* c_k \mathcal{N}_{jk}$$

- Using the fact that the expression for complexity is manifestly symmetric under the exchange of j and k

$$\mathcal{C}(t) = -2 \sum_{n=0}^{\mathcal{D}_\psi - 1} \frac{n}{(b_n!)^2} \sum_{j\neq k}^{\mathcal{D}_\psi} \sin^2(\frac{\omega_{jk}}{2} t) |c_j|^2 |c_k|^2 g_n(E_j) g_n(E_k)$$

Time Evolution of Krylov Complexity

- we arrange the energy eigenvalues in ascending order, $E_1 < E_2 < \cdots < E_{\mathcal{D}_\psi}$, and reformulate the summation

$$\mathcal{C}(t) = -4 \sum_{n=0}^{\mathcal{D}_\psi - 1} \frac{n}{(b_n!)^2} \sum_{\ell=1}^{\mathcal{D}_\psi - 1} \sum_{j=1}^{\mathcal{D}_\psi - \ell} \sin^2\left(\frac{s_{j+\ell}}{2} t\right) |c_{j+\ell}|^2 |c_j|^2 g_n(E_{j+\ell}) g_n(E_j)$$

$$s_{j+\ell} = E_{j+\ell} - E_j$$

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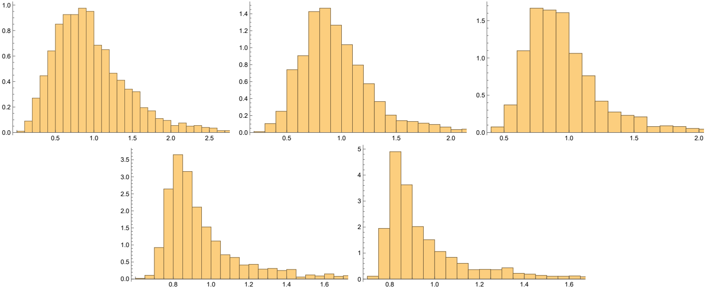
$$s_{j+\ell} = E_{j+\ell} - E_j$$

This is the point where information about the system's nature enters into the computations

Higher Order Level Spacing

- Level spacing for orders 1, 2, 3, 10, and 20 in a chaotic model

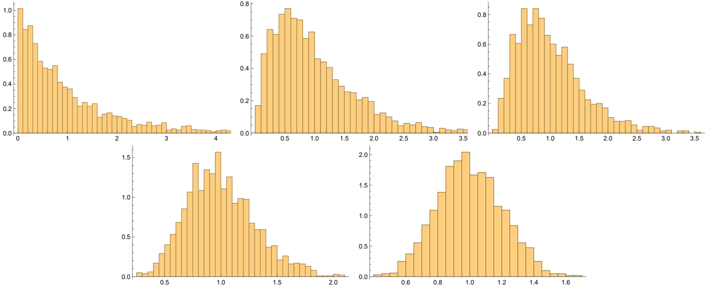


- In chaotic systems, the distribution of s_{j+1} exhibits a peak around a value which is of order one , and this behaviour extends to other ℓ as well.
- Higher order level spacing in chaotic model (Rao,2024)

$$P_n\left(s\right) \propto s^{\alpha} e^{-A(\alpha)s^2}, \quad \alpha = \frac{n\left(n+1\right)}{2}\nu + n - 1$$

Higher Order Level Spacing

- Level spacing for orders 1, 2, 3, 10, and 20 in a integrable model



- For integrable models, the Poisson distribution indicates that the peak of level spacing for s_{j+1} is centered around zero.
- For larger values of ℓ this peak shifts towards an $\mathcal{O}(1)$ value
- Higher order level spacing in integrable model (Rao,2024)

$$P_n(s) = \frac{n^n}{(n-1)!} s^{n-1} e^{-ns} \tag{1}$$

Time Evolution of Krylov Complexity

- In the chaotic model, for all orders, the $\sin^2$ factor oscillates between zero and one, and for large t, its average value approaches $\frac{1}{2}$

$$\overline{\mathcal{C}} \approx -\sum_{n=0}^{\mathcal{D}_\psi - 1} \frac{n}{(b_n!)^2} \sum_{j\neq k}^{\mathcal{D}_\psi} |c_j|^2 |c_k|^2 g_n(E_j) g_n(E_k)\,.$$

- In an integrable model, we decompose the complexity into two parts: the first part involves fast oscillations from $s_{j+\ell}$ for $\ell > 1$, and the second part corresponds to $\ell = 1$

$$\mathcal{C}_0(t) = -4\sum_{n=0}^{\mathcal{D}_\psi - 1} \frac{n}{(b_n!)^2} \sum_{\ell=2}^{\mathcal{D}_\psi - 1} \sum_{j=1}^{\mathcal{D}_\psi - \ell} \sin^2\left(\frac{s_{j+\ell}}{2} t\right) |c_{j+\ell}|^2 |c_j|^2 g_n(E_{j+\ell}) g_n(E_j)$$

$$\mathcal{C}_1(t) = -4\sum_{n=0}^{\mathcal{D}_\psi - 1} \frac{n}{(b_n!)^2} \sum_{j=1}^{\mathcal{D}_\psi - 1} \sin^2\left(\frac{s_{j+1}}{2} t\right) |c_{j+1}|^2 |c_j|^2 g_n(E_{j+1}) g_n(E_j)$$

Time Evolution of Krylov Complexity

- For sufficiently large t, we can estimate the $\sin^2$ of both terms as $\frac{1}{2}$, leading to complexity saturation approaching the infinite time average
- For intermediate (yet still large) t, the second part oscillates
- Since the first term does not consider $\ell = 1$, the resultant constant is smaller than the infinite time average of the complexity $\mathcal{C}_0 < \overline{\mathcal{C}}$

$$\mathcal{C}(t) \approx \mathcal{C}_0 - 2 \sum_{n=0}^{\mathcal{D}_\psi - 1} \frac{n}{(b_n!)^2} \sum_{j=1}^{\mathcal{D}_\psi - 1} \sin^2\left(\frac{s_{j+1}}{2} t\right) |c_{j+1}|^2 |c_j|^2 g_n(E_{j+1}) g_n(E_j)$$

This indicates that, for integrable models, complexity saturation remains below the infinite time average and gradually approaches it from below

Time Evolution of Krylov Complexity

- A distinguishing signature of whether a model is chaotic or integrable can be identified by observing how the Krylov complexity approaches its infinite time average at late times
- For integrable models, the complexity approaches the average from below
- In chaotic models, complexity reaches the saturation value within a finite time
- Depending on the initial state, complexity may also exhibit a peak before reaching saturation in chaotic models

$$\boxed{\begin{aligned} &\mathcal{C}_S \approx \mathrm{Tr}(\rho_{\mathrm{DE}}\mathcal{N}), \qquad \text{for chaotic systems} \\ &\mathcal{C}_S < \mathrm{Tr}(\rho_{\mathrm{DE}}\mathcal{N}), \qquad \text{for integrable systems} \end{aligned}}$$

Explicit Example

The Model

- We consider Ising model

$$H = -J \sum_{i=1}^{N-1} \sigma_i^z \sigma_{i+1}^z - \sum_{i=1}^{N} (g\sigma_i^x + h\sigma_i^z) \,.$$

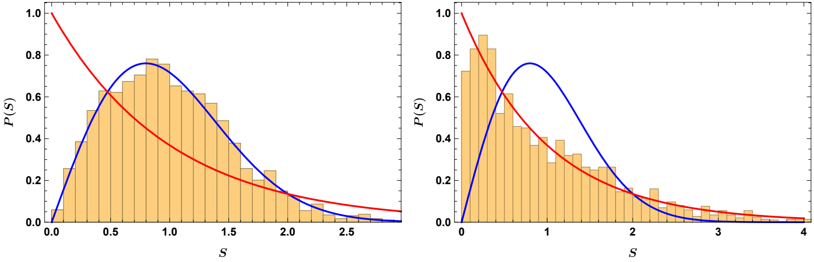
- J, g and h are constants which control behavior of model
- We consider an arbitrary initial state in the Bloch sphere which parameterized by two angles θ and ϕ

$$|\theta, \phi\rangle = \prod_{i=1}^{N} \left(\cos\frac{\theta}{2} \; |Z+\rangle_i + e^{i\phi} \sin\frac{\theta}{2} \; |Z-\rangle_i \right)$$

- We consider three initial states where spins are aligned along the x, y, and z directions, denoted as $|X+\rangle$, $|Y+\rangle$, and $|Z+\rangle$.

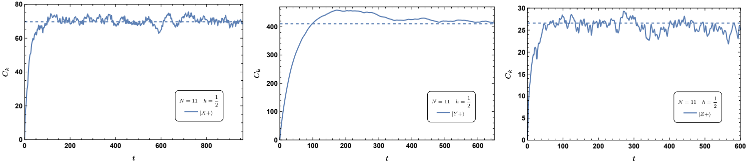
Level spacing

- We will set $h = 0.5$, $g = -1.05$ in chaotic case.
- We will set $h = 0$, $g = -1.05$ in integrable case.

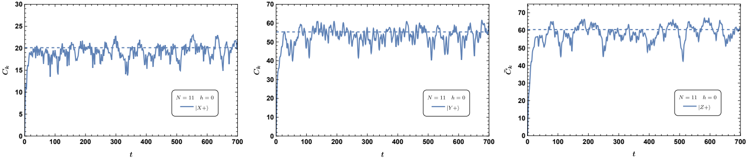


Numerical Result

- Time evolution of complexity in chaotic model



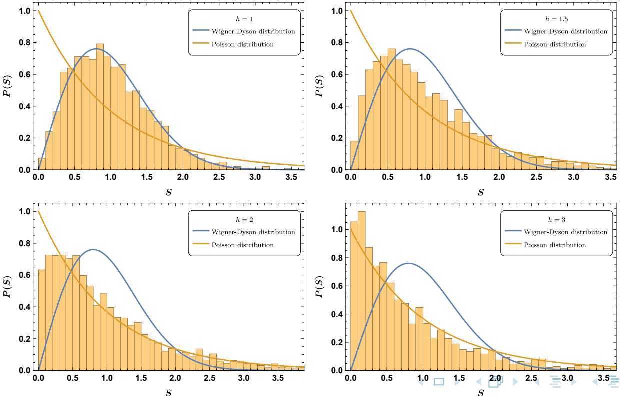
- Time evolution of complexity in integrable model



- In chaotic systems, complexity reaches the saturation value given by the infinite time average at the saturation time.
- In integrable systems, complexity approaches infinite time average value from below over large times
- For a generic initial state, complexity may or may not exhibit the peak

Numerical Result

- Previous numerical computations have focused on two specific points within the model's parameter space correspond to integrable and nearly maximally chaotic
- It is valuable to explore additional points and compare the Krylov complexity and level spacing



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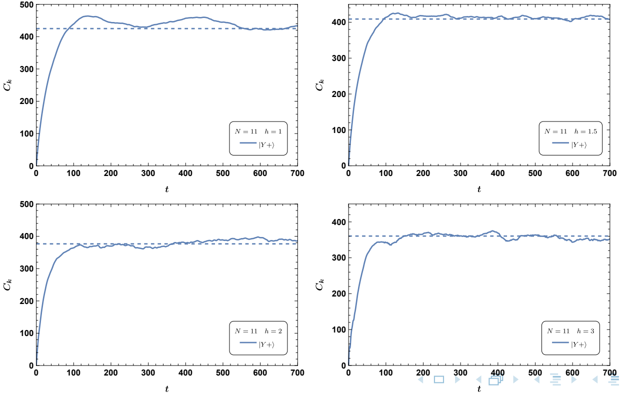

Conclusion

- We have studied Chos for a quantum system using the Krylov basis and Krylov complexity
- Krylov complexity provides a universal tool to study chaos in quantum systems
- Krylov complexity distinguishes chaotic systems and integrable system via saturation dynamic
  1. Chaotic systems: Saturation occurs at a finite time and may exhibit a peak before stabilizing
  2. Integrable systems: Saturation occurs at longer times and approaches the infinite time average from below
- The Ising model supports theoretical predictions

Thank You