Conformal Bootstrap: A Brief Introduction

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Abstract

Conformal bootstraps have emerged as a robust framework for studying the fundamental principles of quantum field theories (QFTs) with conformal symmetry. This article provides an introductory overview of the conformal bootstrap approach, focusing on its theoretical foundations, key concepts, and applications. We discuss the use of symmetry, unitarity, and crossing symmetry to derive constraints on operator dimensions, operator product expansion coefficients, and correlation functions. Furthermore, we explore the numerical methods employed in conformal bootstrapping and highlight their relevance in understanding critical behavior, classifying QFTs, and establishing connections to the AdS/CFT correspondence.

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Contents

1 Introduction

The conformal bootstrap is a non-perturbative mathematical method to constrain and solve conformal field theories, i.e. models of particle physics or statistical physics that exhibit similar properties at different levels of resolution. The analytic conformal bootstrap is an array of techniques to characterize, constrain, and solve strongly interacting quantum field theories using symmetries, causality, unitarity, and other general principles [1]. In the last decade, it has been used to solve conformal field theories at large spin; to place bounds on energy distributions, event shapes, operator product coefficients, and other observables; and to understand aspects of quantum gravity in AdS space [1]. For useful reviews and introduction to conformal field theory and bootstraps see [2, 3, 4, 5].

When using conformal bootstrap methods, several constraints need to be considered to ensure the validity and applicability of the approach. These constraints include:
1. Conformal Symmetry: Conformal bootstrap methods are specifically designed for conformal field theories (CFTs), which possess a higher degree of symmetry known as conformal symmetry. It is essential to ensure that the theory under study or the quantities being analyzed exhibit conformal symmetry. Deviations from conformal symmetry may invalidate the bootstrap approach.
2. Unitarity: Unitarity is a fundamental requirement for any sensible quantum field theory. In the context of conformal bootstrap, it imposes constraints on the scaling dimensions of operators. Unitarity bounds the range of allowed dimensions and excludes regions that lead to unphysical or inconsistent results.
3. Crossing Symmetry: It relates different channels of correlation functions and imposes constraints on the operator spectrum and their operator product expansion coefficients. Crossing equations need to be satisfied to ensure the consistency of the bootstrap analysis.
4. Operator Spectrum: Constraining and determining the operator spectrum is a key aspect of the analysis. Knowledge or assumptions about the operator, such as the existence of specific operators or operator families, can affect the results and interpretation.
5. Computational Resources: When dealing with large operator spaces and high-precision numerical calculations, the feasibility of the analysis must be considered, taking into account available computational power and time constraints.
6. Higher-Dimensional CFTs: Conformal bootstrap techniques are most well-developed and understood in two dimensions and have been successfully applied in the study of critical phenomena. Additional constraints may become necessary for extending the methods to higher dimensions .
7. Approximations and Truncations: In practice, due to computational limitations or analytical difficulties, approximations and truncations are often employed in conformal bootstrap analyses. These approximations introduce potential sources of systematic error, and their impact on the results should be carefully assessed.
It is important to consider these constraints and limitations when applying conformal bootstrap methods to ensure that the results obtained are reliable and consistent.

The conformal bootstrap seeks to constrain and solve CFTs using nonperturbative structures like symmetry, causality, and unitarity. Solve means finding observables, correlation functions, and the spectrum explicitly. In the 80s, bootstrap methods were applied by

Belavin, Polyakov, and Zamolodchikov to exactly solve an infinite class of 2d CFTs [6]. Around 2008, it is developed for higher dimensions, and huge progress was made by numerical methods. The benefits of using conformal bootstrap are ordered as follows:

(1) Studying non-perturbative theories: Quantum chromodynamics (without matter) is described by Yang-Mills theory

$$\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} \tag{1.1}$$

It has a negative beta function and the IR scale of energy grows exponentially with UV coupling g_{UV} as

$$\Lambda_{IR} = \Lambda_{UV} \; exp(-cg_{UV}^2) \tag{1.2}$$

where c is a positive constant. This theory is strongly coupled at large distances and one can explore this theory at IR through lattice QCD or AdS/CFT correspondence. The latter is a dictionary between pure gravity in Anti-de Sitter space and the IR CFT living on the boundary of this space. Another efficient way is to use conformal Bootstrap to compare and match the outcome with the experiments and other theoretical methods. Another example is ϕ^4 theory in $d = 3$:

$$S = \int d^3x \left(\frac{1}{2} (\partial\phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4 \right) \tag{1.3}$$

which is written in Euclidean signature. This theory is free in the UV since m and g have mass dimension 1 and can be ignored at high energies. But strongly coupled at IR. Hence all perturbative approaches fail to exploit the emergent symmetries of the IR theory. Bootstrap only uses nonperturbative structures and thus has a hope of working for strongly-coupled theories.

(2) Studying universal features of QFTs: By studying CFTs, it is possible to map different endpoints of RG flows, and thus understand the space of QFTs. For example, in $d = 3$ the critical point of boiling water, the Ising Model, and the ϕ^4 theory behave differently in the UV limit CFT. But in the IR CFT, they behave in the same fashion with the same fixed point. As a significant result, this point can be found by numerical Bootstrap which agrees with the analytical calculation of quantum field theory. We will explain this example in more detail in sec.(4.3).

(3) New theoretical approach: Performing the bootstrap method does not need a Lagrangian to start with. It actually uses a unique rather abstractive way to extract the data from quantum theories. Therefore, it is useful to study M-theory and quantum gravity in this framework.

In general, Bootstrap in computer science is a technique of loading a program into a computer by means of a few initial instructions which enable the introduction of the rest of the program from an input device. The bootstrap method uses consistency conditions to fix as much freedom as possible in CFT data such as the scaling dimension and OPE coefficients. It is primarily motivated by the desire to understand and explore the fundamental principles underlying quantum field theories (QFTs) and their critical behavior. Here are some reasons why people are interested in conformal bootstraps:

- Understanding CFTs: Conformal field theories are special types of quantum field theories that possess a higher degree of symmetry known as conformal symmetry. They have applications in various areas of physics, such as high-energy physics, condensed matter physics, and statistical mechanics. Conformal bootstrapping provides a systematic framework to study and characterize CFTs by imposing consistency conditions on their correlation functions.

- Nonperturbative Methods: Conformal bootstrapping provides a nonperturbative approach to studying quantum field theories. Unlike traditional perturbative methods that rely on expanding the theory around a weakly coupled regime, the bootstrap approach does not require a priori knowledge of a Lagrangian or a small coupling parameter. Instead, it utilizes the principles of symmetry, unitarity, and crossing symmetry to constrain the theory and obtain nonperturbative results.

- Exploring the Space of QFTs: Conformal bootstrapping allows researchers to explore the space of possible quantum field theories and understand the constraints imposed by fundamental principles. By considering consistency conditions and symmetries, researchers can derive constraints on operator dimensions, operator product expansion coefficients, and other properties of the theory. This helps in identifying and classifying different QFTs and their universality classes.

- Numerical Approaches: Conformal bootstrapping often employs numerical techniques to study correlation functions and extract information about the underlying QFT. These numerical methods, such as the use of conformal blocks, crossing equations, and numerical optimization algorithms, allow for the exploration of theories that are difficult to study analytically. The use of numerical methods also facilitates comparisons with experimental data and numerical simulations in other fields. We discuss these approaches in (4.3).

- Applications in AdS/CFT Correspondence: Conformal bootstrapping has connections to the AdS/CFT correspondence, a duality between certain conformal field theories and gravitational theories in higher-dimensional anti-de Sitter (AdS) spaces. By understanding the conformal bootstrap in the boundary CFT, researchers can gain insights into the corresponding gravitational theory in the bulk, leading to advancements in both fields. We will give some examples of this in sec.(5).

Overall, conformal bootstrapping offers a powerful and versatile framework for studying the properties of quantum field theories, shedding light on their fundamental principles, and exploring the space of possible theories. It combines the power of symmetries, consistency conditions, and numerical techniques to extract nonperturbative information, making it an attractive approach for theoretical physics and mathematics researchers.

To motivate the reader to gain insight into conformal field theories, We will investigate the importance of conformal symmetry and bootstrap program by introducing famous models in statistical field theories in the next section (see [7] for more details).

2 Motivation: Phase transitions and RG flow

The idea of phase transition is a good starting point to get insight of universal behavior. As a liquid of water changes into a gas it possess a phase transition. The phase diagram of water is dipicted in (1). At the critical temperature T_c, the liquid and gas phases becomes indistiguishable. This is a *critical point* where the heat capacity diverges as

$$c \sim \frac{1}{|T - T_c|^\alpha} \tag{2.1}$$

the exponent α is not known precisely, and its approximate value is 0.11008. It is thought not to be a rational number, but should instead be viewed as a universal mathematical constant, similar to π or e, but more subtle. Remarkably, the same exponent occurs for all gases. It also occurs in other systems, including a certain class of magnets. It's as if all knowledge of the microscopic physics has been washed away, leaving us with something pure, that carries only a vague memory of what lies underneath. This phenomenon is known as *universality*. This paradigm revolves around two deep facts about the Universe we inhabit: Nature is organized by symmetry and scale.
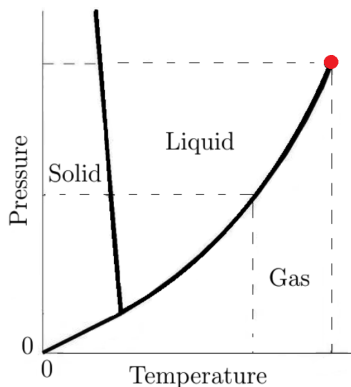

Figure 1: Phase diagram of water. The Intersection of three lines is the triple point where three water phases coexist in thermodynamic equilibrium. The line between liquid and gas phases ends on a critical point (red dot) with temperature T_c while the other one goes forever.

Nature is organized by symmetry: Can be confirmed mostly by studying Landua theory (see sec.(2.2)) where different phases of matter are characterized by symmetry. For example, a solid form of a lattice with magnetization m has a $\mathbb{Z}_2$ symmetry since if all the spin configurations change from -1 to $+1$ the physics does not change. Deforming the lattice into liquid does not necessarily preserve this and the discrete translation symmetry will break.

Nature is organized by scale: In the 1970s a mathematical formalism was developed that makes these ideas concrete. This formalism, which is studied by Wilson, Kadanoff, and others is called the renormalization group and provides a framework to describe physics at different scales. The renormalization group gets little coverage in popular science articles, yet is arguably the single most important advance in theoretical physics in the past 50 years. We will explain the details in sec.(2.4).

2.1 Ising Model

We consider a lattice in d spatial dimensions, containing N sites. On each lattice site $i = 1, \ldots, N$, there lives a discrete variable that can take one of two states: $s_i = +1$ or $s_i = -1$. It is useful to refer to these states as spins, with $s_i = +1$ corresponding to spin up, and $s_i = -1$ to spin down. However, we won't be using any machinery from quantum mechanics: there are no non-commuting operators, nor quantum superpositions of states. In this sense, the Ising model, while discrete, is purely classical. The collection of spins $\{s_i\}$ has energy

$$E = -B \sum_i s_i - J \sum_{\langle ij \rangle} s_i s_j \tag{2.2}$$

The first term arises due to an external magnetic field, B that we impose on the system. It has the effect that, for $B > 0$, the spins want to be up because that will lower their energy. The notation $\langle ij \rangle$ means that we sum over all "nearest neighbor" pairs in the lattice. The number of such pairs depends both on the dimension d and the type of the lattice. J is the coupling between nearest-neighbor pairs. If $J > 0$, neighboring spins prefer to be aligned ($\uparrow\uparrow$ or $\downarrow\downarrow$). In the context of magnetism, such a system is called a ferromagnet. If $J < 0$, the spins want to anti-align ($\uparrow\downarrow$). This is an anti-ferromagnet. In the following, we choose $J > 0$ although, for our purposes, the differences are minor. The emergence of this behavior is coming from a $\mathbb{Z}_2$ symmetry.

This energy discussed above is at zero temperature. We are interested in the physics of the Ising model at a finite temperature T. We can already get some intuition for what will happen. The interaction energy encourages the spins to align in the same way. The magnetic field encourages the spins to align in a particular way. Meanwhile, the temperature encourages the spins to ignore both the interactions and magnetic field because, at finite temperatures, energy is no longer at a premium. Instead, entropy S becomes more important. Since there are many more random configurations than aligned configurations, the temperature will tend to mess up the nice ordered states that the interactions and magnetic field have so carefully prepared. So it is convenient to study the free energy $F = E - TS$ and see when this energy minimizes. In other words, We want to maximize the entropy in which sense the spins point randomly. In this case, the partition function of the system is defined as:

$$Z = \sum_{\{s_i\}} e^{-\beta E[s_i]} \tag{2.3}$$

where β is the inverse of temperature T. Of particular interest to us will be the average spin of the configuration, which we refer to as the equilibrium magnetization and can be written as

$$m = \frac{1}{N} \left\langle \sum s_i \right\rangle = \frac{1}{N\beta} \frac{\partial logZ}{\partial B} \tag{2.4}$$

It turns out that the sum is straightforward in a $d = 1$-dimensional lattice. An exact solution also exists in $d = 2$ when $B = 0$, originally due to Onsager. It is not straightforward. In higher dimensions, no exact solutions are available, although various expansions and

tricks have been invented to manipulate the sum to extract some interesting information. The energy of the Ising model (2.2) provides the microscopic description of our system in terms of individual spins which is more likely a fine-grained configuration. Computing the partition function exactly gives us a macroscopic description of the system in terms of thermodynamic variables like temperature T and magnetic field B. What we're looking for is something in between. We want to go to the regime where the partition function is more like a coarse-grained configuration with dependence on macroscopic variables B, J, T not on spin. In this case, the macroscopic quantity that we want to focus on is the magnetization $-1 \leq m \leq 1$ instead of s_i. Note that here any spin configurations can have the same magnetization. This is a fact of the coarse-graining of information in the system. The partition function in this case is

$$Z(B, J, T) = \sum_m e^{-\beta F(m)} = \frac{N}{2} \int_{-1}^{1} dm\, e^{-\beta F(m)} \tag{2.5}$$

By assertion, we assume that for a sum of magnetizations, we provide another sum over spins that leads to the specific magnetization in the first sum. $F(m)$ is called the effective free energy. In the second equality, we are assumed that the number of sits N is very large and the amount of discrete value m is approximately continuous due to a sum of large spin configurations. The factor of $\frac{N}{2}$ is the (inverse) width between the allowed m values. In the next step, we use the mean-field approximation approach which says instead of summing over spins in (2.2) we rather sum over magnetizations. We wish to sum over configurations $\{si\}$ with $\sum_i s_i = Nm$. We can get an estimate for the energy of such configurations by replacing each spin s in (2.2) with its expectation (i.e. mean) value $\langle s \rangle = m$,:

$$E = -B \sum_i m - J \sum_{\langle ij \rangle} m^2 \tag{2.6}$$

the sums give an overall factor of N which we can pull out and finally:

$$\frac{E}{N} = -Bm - \frac{1}{2}Jqm^2 \tag{2.7}$$

where q is the number of nearest neighbors and depends on the dimension of the lattice. The effective free energy per the number of unit spin turns out to be:

$$f(m) = \frac{F(m)}{N} \approx -Bm - \frac{1}{2}Jqm^2 - T(log2 - \frac{1}{2}(1+m)log(1+m) - \frac{1}{2}(1-m)log(1-m)) \tag{2.8}$$

At equilibrium:

$$\frac{\partial f}{\partial m} = 0 \quad \implies \quad m = tanh(\beta B + \beta Jqm) \tag{2.9}$$

There is a nice intuition behind this equation. It can be derived by assuming that each spin experiences an effective magnetic field given by $B_{\text{eff}} = B + Jqm$, which includes an extra contribution from the spins around it. In this way, the tricky interactions in the Ising model have been replaced by an averaged effective field B_{eff}. This is sometimes called the mean field and gives its name to this technique.

2.2 Landau Theory

Landau's approach focuses on situations where the order of m is small. Here we can Taylor expand (2.8),

$$f(m) \simeq -Bm + \frac{1}{2}(T - Jq)m^2 + \frac{1}{12}Tm^4, \tag{2.10}$$

where we have dropped away the orders higher than m^4 and ignored the term $-Tlog2$ since it does not play an important role in our final results. Let us consider $B = 0$. In this case, there are two possibilities (see fig.(2)). We can denote $Jq = T_c$. First, when $T > T_c$ we have one minimum for free energy at $m = 0$ which already tells us that the temperature is high enough to disorder the system. Second, when $T < T_c$, the magnetization would be preserved in the system and minimize the free energy at $m = +1$ (for aligned spin configurations) and at $m = +1$ (for anti-aligned spin configurations). At $T = T_c$ the system experiences a phase transition, changing the system from phase $m = 0$ to $m \neq 0$ (fig. (3)).
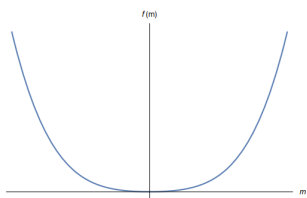

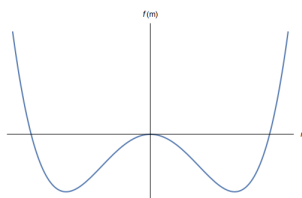**Figure 1:** Free energy when $T > T_c$ **Figure 2:** Free energy when $T < T_c$

Figure 2: Large temperatures can disorder the set of spin orientations and free energy takes a global minimum (fig.1). At low temperatures, the magnetic field outweighs the contribution of ordering the spins in two aligned and anti-aligned sets, resulting in two global minimums and one local maximum (fig. b). The system possesses a phase transition at $T = T_c$. Figure taken from [7].
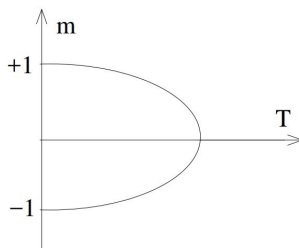

Figure 3: When $T < T_c$, the magnetization can take two different values. They emerge as a single value in a certain temperature T_c. Figure from [7].

The free energy (2.10) is invariant under the $\mathbb{Z}_2$ symmetry $m \rightarrow -m$. This is no coincidence: it follows because our microscopic definition of the Ising model (2.2) also enjoys this symmetry when $B = 0$. However, below T_c, the system must pick one of the two ground states $m = +m_0$ or $m = -m_0$. Whichever choice it makes breaks the $\mathbb{Z}_2$ symmetry. When

a symmetry of a system is not respected by the ground state we say that the symmetry is spontaneously broken. This will become an important theme for us as we move through the course. It is also an idea that plays an important role in particle physics.

2.3 Landau-Ginzburg theory

Landau-Ginzburg theory is the generalization of Landau theory in which we consider magnetization as a field that depends locally on space: $m(x)$. We put several sites into a box of length a and repeat this structure as a mesh design applied on a lattice (fig.(4)). The space-time component x is the location of each box on the lattice. The magnitization of each box is $m(x) = \frac{1}{N_{\text{box}}} \sum s_i$. In general, there will be many spin configurations for each m(x). We then sum over all possible values of m(x). The Partition function hence:
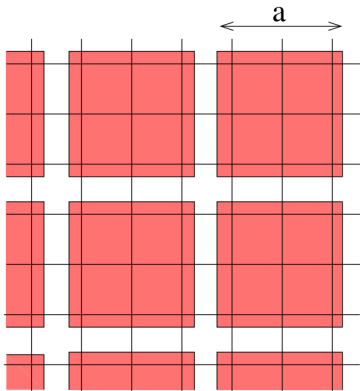

Figure 4: A lattice is constructed by a set of cubic boxes with edges of length a and N_{box} number of spins. Magnetization of each box $m(x)$ depends on the position of the center of the box from the center of lattice x.

$$Z = \int \mathcal{D}m(x) e^{\beta F[m(x)]} \qquad (2.11)$$

the magnetization here can be thought of as a field. We demand that the theory have the following properties:

Locality: The nearest neighbor interactions of the Ising model mean that a spin on one site does not directly affect a spin on a far-flung site. It only does so through the intermediate spins. The same should be true of the magnetization field $m(x)$. The result is that the free energy should take the form

$$F[m(x)] = \int d^d x f[m(x)] \qquad (2.12)$$

where $f[m(x)]$ is a local function. It can depend on $m(x)$, but also on $\nabla m(x)$ and higher derivatives. These gradient terms control how the field $m(x)$ at one point affects the field at neighboring points.

Translation and Rotation invariance: The original lattice has a discrete translation symmetry. For certain lattices (e.g. a square lattice) there can be discrete rotation symmetries.

At distances much larger than the lattice scale, we expect the continuum version of both these symmetries emerges, and our free energy should be invariant under them.

$\mathbb{Z}_2$ symmetry: When $B = 0$, the original Ising model (2.2) is invariant under the symmetry $s_i \to -s_i$, acting simultaneously on all sites. This symmetry is inherited in our coarse-grained description which should be invariant under $m(x) \to -m(x)$. When $B \neq 0$, the Ising model is invariant under $m(x) \to -m(x)$, together with $B \to -B$. Again, our free energy should inherit this symmetry.

Analyticity: We will make one last assumption: that the free energy density is an analytic function of $m(x)$ and its derivatives. Our primary interest lies in the critical point where m first becomes non-zero, although we will also use this formalism to describe first-order phase transitions where $m(x)$ is small. In both cases, we can Taylor expand the free energy in m and restrict attention to low powers of m.

With this assumptions in hand, the effective free energy takes the general form:

$$F[m(x)] = \int d^d x \left(\frac{1}{2} \gamma(T)(\nabla m)^2 + \frac{1}{2} \alpha_2(T) m^2 + \frac{1}{4} \alpha_4(T) m^4 + \dots \right), \qquad (2.13)$$

From now on we use the notation $\phi(x)$ instead of $m(x)$. We terminate the higher orders of (2.13) and stick to square term and. We also add a linear source, therefore:

$$F[m(x)] = \int d^d x \left(\frac{1}{2} \gamma(\nabla \phi)^2 + \frac{1}{2} \mu^2 \phi^2 - B\phi \right) \qquad (2.14)$$

by using quantum field theory tricks the correlation function

$$\langle \phi(x)\phi(y) \rangle = \frac{1}{\beta^2} \left[\frac{\delta^2 logZ}{\delta B(x) \delta B(y)} \right]_{B=0} \qquad (2.15)$$

can be computed by the partition function and it takes the asymptotic forms:

$$\langle \phi(x)\phi(y) \rangle \sim \begin{cases} \frac{1}{r^{d-2}}, & r << \xi \\ \frac{e^{-r/\xi}}{\xi^{d/2-3/2} r^{(d-1)/2}}, & r >> \xi \end{cases} \qquad (2.16)$$

This is known as the Ornstein-Zernicke correlation. We have introduced a length scale parameter ξ

$$\xi = \frac{\sqrt{\gamma}}{\mu} \qquad (2.17)$$

which is called the correlation length. ξ provides a characteristic length scale for the fluctuations. You can find out from previous discussions that $\mu^2 \sim |T - T_c|$. Therefore: $\xi \sim 1/|T - T_c|^{1/2}$. You can see if $T \to T_c$ or equivalently $\xi \to \infty$, we reach a critical point in which the correlation function (2.16) becomes scale invariant. This is telling us that the system will undergo fluctuations of arbitrarily large size. This is the essence of a second-order phase transition.

2.4 Renormalization Group

We have to ask the following question: at what scale is this free energy describing our system? or What scale of physics am I interested in?

We took a cluster of spins and look at quantities defined for these clusters in Landaue-Ginzburg theory, but then we never specified how closely we are looking at the system as observers. We as an experimentalist, need to have a nice resolution over this system of spins. It is conceivable that the theory we will write down will depend on how fine resolution we can get to. In order to do this we have to use the idea of a renormalization group. If our experiment is only concerned with physics that only is sensitive to modes below a certain energy scale, then it makes no more sense for us to even be discussing the energy modes above that scale. They are completely fixed and hence non-dynamical. The idea of the renormalization group is then to say that the scale at which we observe the system, has relevance to the physics question that we can ask and the effective field theory depends on that scale.

After rescaling ϕ in (2.14), the free energy with quadratic and other terms is

$$F[\phi(x)] = \int d^d x \left(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\mu^2 + \frac{1}{4!}g\phi^4 + \dots \right) \tag{2.18}$$

We can set an energy scale cut-off Λ on the theory. Suppose we care about physics up to a length scale L. Then we have no real interest in the Fourier modes ϕ_k with $k >> \Lambda \sim 1/L$. This suggests that we can write down a different theory, that has a lower cut-off, $\Lambda' = \Lambda/\zeta$ with $\zeta > 1$. We defined our theory at Λ and we can do computations to find the low energy effects. Above Λ our momentum modes are frozen to zero. Equivalently, one can define the theory at Λ'. The question is does either theory described by the same theory? To answer this we write our Fourier modes of ϕ as:

$$\phi_k = \phi_k^- + \phi_k^+ \tag{2.19}$$

where ϕ_k^- describe the long-wavelength fluctuations and ϕ_k^+ describe the short-wavelength fluctuations that we don't care about

$$\phi_k^- = \begin{cases} \phi_k, & k < \Lambda' \\ 0, & k > \Lambda' \end{cases}, \qquad \phi_k^+ = \begin{cases} \phi_k, & \Lambda' < k < \Lambda \\ 0, & \text{otherwise} \end{cases} \tag{2.20}$$

Now by doing these three steps of what is known as the renormalization group (RG): 1- Integrate out high-k modes. 2- Rescale momentum. 3- Rescale fields appropriately. One finds out that the two theories are related such that the fields are rescaled and the coefficients like the coupling constant, have a running dependent behavior on ζ:

$$F_\zeta[\phi(x)] = \int d^d x \left(\frac{1}{2}(\nabla\phi')^2 + \frac{1}{2}\mu^2(\zeta)\phi' + \frac{1}{4!}g(\zeta)\phi'^4 + \dots \right) \tag{2.21}$$

We see that this procedure induces a continuous map from $\zeta \in [1, \infty)$ onto the space of coupling constants. Our original coupling constants in (2.18) are those evaluated at $\zeta = 1$. We say that the coupling constants flow, where the direction of the flow is telling us what the couplings look like on longer and longer length scales. These flows are known as *renormalization group flow* or *RG flow*. The equations which describe these flows are known as beta functions.

3 Conformal Field Theory

Conformal field theories (CFTs) are quantum field theories (QFTs) that are invariant under the conformal group, which consists of changes in coordinates that preserve angles. These include translations and Lorentz transformations, that define the Poincare group of standard relativistic QFT, as well as rescalings, and special conformal transformations. It has several applications, from the definition of QFTs to phase transitions in condensed matter, to holography and quantum gravity. Four-dimensional $\mathcal{N} = 4$ super Yang-Mills, 2d and 3d Ising models, and critical $O(N)$ models are a few examples of CFTs. Most scale-invariant QFTs are also conformally invariant, so even though the conformal group is strictly bigger, it will guide our intuition to think of CFTs as scale-invariant QFTs. As we argued earlier, QFTs are defined as renormalization group (RG) flows from a microscopic theory at short distance scales (the UV), to a theory at long distance scales (the IR):
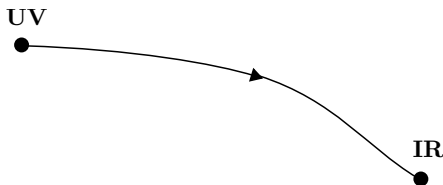
There are three possibilities for IR phases: (a) a theory with a mass gap, (b) a theory with massless particles in the IR, and (c) a Scale Invariant (SI) theory with a continuous spectrum. It is the last class that we will call CFT and will mostly study in this paper. Many theories in the UV phase can turn into one scale-invariant IR theory, a one CFT. Let us consider a flat space in d dimensions and transformations $x_\mu \to x'_\mu(x)$ thereof which locally preserve the angle between any two lines, such that:

$$(ds')^2 = \Omega(x)^2 \ (ds)^2 \tag{3.1}$$

Such transformations are called *conformal transformations*. $\Omega(x)$ is scale or conformal factor which for flat spaces $\Omega(x) = 1$ corresponds to the Poincare group. This is for general transformations. However, one can stick to the infinitesimal transformations:

$$x'_\mu = x_\mu + v_\mu(x) \tag{3.2}$$
$$\Omega(x) = 1 + \sigma(x)$$

which indeed covers the same algebraic conformal structure. With little work with these transformations (3.3) and (3.1), one can find the infinitesimal form of conformal transformations as:

$$v_\mu(x) = a_\mu - \omega_{[\mu\nu]}x^\nu + \kappa x_\mu + (b_\mu x^2 - 2x_\mu b_\nu x^\nu) \tag{3.3}$$

The first term shows the infinitesimal form of translation. The other three terms correspond to infinitesimal rotation, dilation, and special conformal transformations respectively.

In two-dimensional CFT, the number of (quasi-) primary fields is infinite and there are examples with continuous spectrum. On the other hand, in $d > 2$, we only have the global

part of conformal symmetry and it is conjectured that any CFT has a discrete spectrum. In any dimension d, the spectrum can be approximately given by[2]:

$$\rho(\Delta) \sim e^{c\Delta^{(d-1)/d}} \tag{3.4}$$

where c is a constant and Δ is the scaling dimension (see next section).

In this paper, we focus on Euclidean CFTs in higher than two dimensions and discuss the numerical bootstrap applied to them. However, We discuss the main differences between 2d CFTs and those in higher dimensions as follows.

Conformal field theories in $d \geq 3$, the symmetry algebra is finite-dimensional and it is the conformal algebra. In 2D CFTs, the symmetry algebra is significantly enhanced and becomes infinite-dimensional. It is known as the *Virasoro algebra*, which is a central extension of the *Witt algebra*. It consists of an infinite tower of generators (modes) that generate conformal transformations. These generators include the stress-energy tensor, which plays a central role in 2D CFTs, as well as infinitely many other primary fields and their descendants (see sec.(3.2)). In 2d, there are extra constraints, modular invariance, and infinite conformal charges. By modular invariance, one can relate all sectors of CFT_2. Using conformal bootstrap in higher dimensions, we can only solve a subsector of CFT. In more technical terms, that is because the algebra of the stress tensor does not close. This is not an issue, and we can efficiently solve ninety-nine percent of the theory.

In 2d CFT the power of the conformal bootstrap was demonstrated in the famous 1984 paper by Belavin, Polyakov, and Zomolodchikov (BPZ) [6]. The 2d conformal algebra (Virasoro algebra) is infinite-dimensional. The generators are called L_n and $\overline{L}_n$. The finite-dimensional subalgebra of global conformal transformations is embedded into it as follows:

$$\underbrace{\ldots, L_{-3}, L_{-2}}_{\text{extra raising operators}}, \underbrace{L_{-1}, \quad L_0, \quad L_1,}_{P_\mu \ \{D, M_{\mu\nu}\} \ K_\mu} \underbrace{L_2, L_3, \ldots}_{\text{extra lowering operators}} \tag{3.5}$$

$$\ldots \quad \overline{L}_{-1}, \quad \overline{L}_0, \quad \overline{L}_1, \quad \ldots$$

The state in this theory, is the *highest weight state* $|h\rangle$ satisfies the conditions $L_0|h\rangle = h|h\rangle$ and $L_n|h\rangle = 0$ for $n > 0$, where L_0 and L_n are the Virasoro generators. The eigenvalue h is known as the conformal weight or scaling dimension[3] of the highest weight state. The lowering operators in the Virasoro algebra are denoted as L_n with $n < 0$. A state $|h\rangle$ is annihilated by these lowering operators, i.e., $L_n|h\rangle = 0$ for $n < 0$.

The highest weight state provides a starting point to construct the entire representation of the Virasoro algebra by acting with the raising operators (L_{-n}) on the highest weight state. These raising operators create new states with higher eigenvalues of L_0, forming a tower of states known as a *Verma module*. More about 2d CFTs can be found in [8].

In this section, we give the basic definitions used in conformal field theory and study conformal algebra to investigate the form of correlation functions. We will explore how to quantize a theory with conformal symmetry and look for possible interactions in such a theory. This section is a necessary introduction to understanding and starting bootstrap technology.

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[2] The reason why Δ represent the spectrum of the theory will become clear in sec.(3.4)

[3] For higher dimensions, we will denote it by Δ.

3.1 Conformal or Scaling Dimension

The massless scalar field action, given by

$$S = \int d^d x \phi(x) \,\Box\phi(x) \tag{3.6}$$

is conformally invariant, because it is invariant under scale transformations:

$$x^\mu \longrightarrow \widetilde{x}^\mu = \lambda x^\mu,$$
$$\phi(x) \longrightarrow \widetilde{\phi}(\widetilde{x}) = \lambda^{-\Delta}\phi(x), \tag{3.7}$$

where λ is the dilation factor, and where Δ is *scaling dimension* of the field ϕ. Indeed, under (3.7), the action takes the form

$$S \longrightarrow \widetilde{S} = \int \lambda^d d^d x \widetilde{\phi}(\widetilde{x}) \,\Box\widetilde{\phi}(\widetilde{x}) \tag{3.8}$$

$$= \int d^d x (\lambda^{d-2-2\Delta})\phi(x) \,\Box\phi(x) \tag{3.9}$$

Now, we check that this action would be scale invariant provided we make the choice

$$\Delta = \frac{d-2}{2} \tag{3.10}$$

which is equal to the naive dimension of the scalar field ϕ. This is not always correct. When we consider interacting terms for a renormalizable theory, we have to write the bare parameters of the free theory (such as the coupling constant) in terms of the physical parameters that appear in counter terms. We leave this as a practice. In that case, we can compute the beta function of the theory:

$$\beta(g) = \frac{dg}{dln\mu} \tag{3.11}$$

where g is the coupling constant of the interacting theory and μ is the dimensionless scale of energy. When $\beta(g_*) = 0$, we are at a fixed point where the theory is independent of the energy scale. If we provide a similar transformations as (3.7), the theory is scale-invariant up to an *anomalous dimension* γ:

$$\Delta = \frac{d-2+\gamma}{2} \tag{3.12}$$

to account for the presence of interaction and rescaling of the coupling constant. From now on when we write Δ we mean scaling dimension (3.12).

As stated in the introduction, at a fixed point of the renormalization group, any scale must be washed away. This is already enough to ensure that correlation functions must take the form of a power law,

$$\langle \phi(x)\phi(0) \rangle \sim \frac{1}{r^{2\Delta}} \tag{3.13}$$

The only freedom that we have is in the choice of exponent which we have chosen to parameterize as γ. One of the tasks of the RG procedure is to compute γ.

3.2 Algebra

In this section, we will express the algebra for any CFT. In the Euclidean signature, the conformal algebra is isomorphic to the algebra of $SO(d+1;1)$. The conformal transformations are:

Translation: $x^\mu \to x'^\mu = x^\mu + a^\mu$,   Rotation: $x^\mu \to x'^\mu = R^\mu{}_\nu x^\nu$

Dilation: $x^\mu \to x'^\mu = \kappa x^\mu$,   Special Conformal: $x^\mu \to x'^\mu = \dfrac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}$

The generators of conformal algebra are translation generators P^μ, rotation generators $M^{\mu\nu}$, dilation generator D, and special conformal generators K^μ. The commutation relations among these generators are:

$$[D, P_\mu] = -iP_\mu \tag{3.14}$$

$$[D, K_\mu] = iK_\mu \tag{3.15}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}) \tag{3.16}$$

$$[M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \tag{3.17}$$

$$[M_{\mu\nu}, K_\rho] = i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu) \tag{3.18}$$

$$[K_\nu, P_\mu] = 2i(\eta_{\mu\nu}D - M_{\mu\nu}) \tag{3.19}$$

Any other commutations that are not written here are zero. These commutation relations capture the algebraic structure of the conformal transformations and govern the symmetries of the conformal field theory.

Let $\mathcal{O}_I(x) \equiv \mathcal{O}^I_{\Delta_I, r}(x)$ be an operator of dimension Δ_I in irreducible representation r and let $x \to x'(x)$ be a conformal transformation. Then for some operators in the CFT:

$$\mathcal{O}_I(x) \to \mathcal{O}'_I(x') = \Omega(x)^{-\Delta} R^J_I(x)\mathcal{O}_J(x) \tag{3.20}$$

where R^J_I is the representation of the operator. For instance, for a scalar field, it is just the identity. The operator that satisfies this condition or equivalently (3.29) is called *primary*. All other operators in the conformal multiplet are called *descendants* and are obtained from the primary by acting $n \geq 1$ times with P_μ, which means that they are simply its derivatives:

$$\mathcal{O}^i_n := P_{\mu_1} \dots P_{\mu_n} \mathcal{O}^i \tag{3.21}$$

Primaries transform homogenously while descendants do not. By homogenous we mean that the operator, after conformal transformation or any other operation, still has one term (itself) with an overall factor.

Applying an infinitesimal translation on the field $\mathcal{O}_I(x + \epsilon)$ and Taylor expanding it will produce the generator $P_\mu \equiv -i\partial_\mu$ on its linear term in ϵ. Repeating this procedure for other conformal transformations and using the fact that any generator does not change the conformal vacuum, would lead to

$$[P_\mu, \mathcal{O}_I(x)] = i\partial_\mu \mathcal{O}_I(x) \tag{3.22}$$

$$[D, \mathcal{O}_I(x)] = -i(x.\partial + \Delta)\mathcal{O}_I(x) \tag{3.23}$$

$$[M^{\mu\nu}, \mathcal{O}_I(x)] = i(x_\mu \partial_\nu - x_\nu \partial_\mu)\mathcal{O}_I(x) - \mathcal{O}_J(x)(S_{\mu\nu})^J_I \tag{3.24}$$

$$[K_\mu, \mathcal{O}_I(x)] = i(x^2 \partial_\mu - 2x_\mu x.\partial - 2\Delta x_\mu)\mathcal{O}_I(x) + 2\mathcal{O}_J(x)(S_{\mu\nu})^J_I x^\nu \tag{3.25}$$

with $S_{\mu\nu}$, finite-dimensional matrices of spin-l representation of rotation $M_{\mu\nu}$. Any operator should have definite scaling dimension Δ and spin l.

It is suitable to study the primary operator inserted at the origin $x = 0$. In this case, the commutation relations are:

$$[P_\mu, \mathcal{O}_I(0)] = i\partial_\mu \mathcal{O}_I(0) \tag{3.26}$$

$$[D, \mathcal{O}_I(0)] = -i\Delta \mathcal{O}_I(0) \tag{3.27}$$

$$[M^{\mu\nu}, \mathcal{O}_I(0)] = -i\mathcal{O}_J(0)(S_{\mu\nu})^J_I \tag{3.28}$$

$$[K_\mu, \mathcal{O}_I(0)] = 0 \tag{3.29}$$

From (3.27), one can find eigenstates for the dilation operator with an eigenvalue equal to the scaling dimension Δ. We will see that this becomes a central key in radial quantization. These generators also satisfy the *conformal Ward identity*:

$$\langle [L_A, \mathcal{O}_1(x_1)] \mathcal{O}_2(x_2) \dots \mathcal{O}_n(x_n) \rangle + \langle \mathcal{O}_1(x_1)[L_A, \mathcal{O}_2(x_2)] \dots \mathcal{O}_n(x_n) \rangle + \dots = 0 \tag{3.30}$$

where L_A are the conformal group generators $P_\mu, K_\mu, D, M_{\mu\nu}$.

3.3 Correlation functions

Consider two scalars $\phi_1(x_1)$ and $\phi_1(x_2)$. The correlation function that is invariant under Poincare transformations turns out to be of a general form

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = cf(x_{12}^2), \qquad x_{ij} = |x_i - x_j| \tag{3.31}$$

requiring this correlation to be invariant under dilatation gives:

$$f(x_{12}^2) = \frac{1}{(x_{12}^2)^{\frac{1}{2}(\Delta_1 + \Delta_2)}} \tag{3.32}$$

Moreover, having this correlation invariant under special conformal transformation gives the constraint $\Delta_1 = \Delta_2$. Therefore the general form of the two-point function in CFT has the form:

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \frac{c\delta_{\Delta_1,\Delta_2}}{x_{12}^{2\Delta_1}} \tag{3.33}$$

Conformal invariance is also powerful enough to fix a three-point function of primary scalars, up to an overall coefficient:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3) \rangle = \frac{\lambda_{123}}{x_{12}^{h_{123}} x_{13}^{h_{132}} x_{23}^{h_{231}}}, \qquad h_{ijk} = \Delta_i + \Delta_j - \Delta_k \tag{3.34}$$

with λ_{123} constant, satisfies the conformal Ward identity. With four points, there are nontrivial conformally invariant combinations of the points called *conformal cross-ratios*:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2} \tag{3.35}$$

higher correlation functions can depend nontrivially on these cross-ratios.

3.4 Radial Quantization

Suppose we have $(d-1)$-dimensional surfaces in space-time, where time evolve from one surface to another (Fig. (3.4) left). Traditionally, when we have Poincare invariant CFTs we choose the generator of this time translations to be the Hamiltonian H or the zeroth component of the momentum. Our states are defined on each of these hypersurfaces. We assign a Hilbert space on this foliations and the state lives on this space. In this space we define the inner product of states, and how we evolve states from one Hilbert space to another by a unitary operator $U = e^{iHt}$.

In CFTs, we have more symmetries than just Poincare and it is convenient to consider another type of foliation of space-time. In CFT we choose a foliation by spheres S^{d-1} of different radii (fig. (3.4), right). The role of the Hamiltonian here is played by the Dilatation operator D that moves us from one state in a hypersphere to another hypersphere in this foliation. States in this Hilbert space are the eigenstates of the dilatation operator. According to (3.27), at $x = 0$, we can define states with specific eigenvalue $-i\Delta$ that the dilatation has as eigenstate. There going to be some state $|\Delta\rangle$ that the dilatation acts on. These will be the states generated by the action of some operator on the vacuum $|0\rangle$:

$$D|\Delta\rangle = D\mathcal{O}_\Delta|0\rangle = [D, \mathcal{O}_\Delta]|0\rangle = -i\Delta\mathcal{O}_\Delta|0\rangle = -i\Delta|\Delta\rangle \tag{3.36}$$

where we used $D|0\rangle = 0$. States are generated on the sphere by inserting operators inside the sphere. Inserting nothing leads to the vacuum state. By inserting $O_\Delta(x = 0)$ we get the state $|\Delta\rangle$. When we insert $O_\Delta(x \neq 0)$, this does not give an eigenstate of the dilatation but an infinite linear combination of states in the Hilbert space:

$$\mathcal{O}_\Delta(x)|0\rangle = e^{iP.x}\mathcal{O}_\Delta(0)e^{-iP.x}|0\rangle = e^{iP.x}|\Delta\rangle = \sum_n \frac{1}{n!}(iP.x)^n|\Delta\rangle \tag{3.37}$$

Now we wnat to see if acting on Δ by P_μ is an eigenstate of the dilatation or not:

$$D(P_\mu|\Delta\rangle) = [D, P_\mu]|\Delta\rangle + P_\mu D|\Delta\rangle = -iP_\mu|\Delta\rangle - i\Delta P_\mu|\Delta\rangle = -i(\Delta + 1)(P_\mu|\Delta\rangle) \tag{3.38}$$

the state is an eigenstate with eigenvalue $-i(\Delta+1)$. So acting on the momentum repeatedly raises the scaling dimension of the state by one unit each time. Doing the same scenario for K_μ would lower the eigenvalue by one unit each time. Therefore, Momentum (P_μ) and special conformal (K_μ) raise/lower the levels and act like ladder operators to generate new states in a representation.

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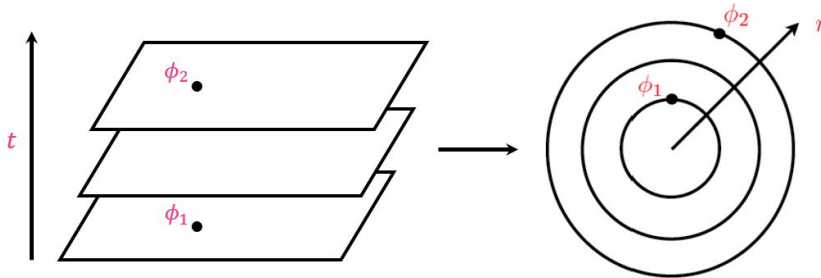
3.5 Unitarity Bound

Unitarity in CFT requires us to know the conjugation of operators to compute norms. For this purpose, It is useful to perform a map from $\mathbb{R}^d$ to $\mathbb{R} \times S^{d-1}$. This maps radial foliations to a cylinder in which every circle describes a sphere S^{d-1} (see figure below). This can be done by the definition $r = e^\tau$ for the radial component of the line element of $\mathbb{R}^d$. τ can be considered as a time component. In that case rescaling $r \to \kappa r$ means time translation $\tau \to \tau + log\kappa$. We can define cylinder operators:

$$\mathcal{O}_{\text{cyl}}(\tau, n) = e^{\Delta \tau} \mathcal{O}_{\text{flat}}(x = e^\tau n), \qquad \mathcal{O}_{\text{cyl}}^\dagger(\tau, n) = \mathcal{O}_{\text{cyl}}(-\tau, n) \tag{3.39}$$

we can translate them back to flat operators:

$$\mathcal{O}_{\text{flat}}^\dagger(x) = e^{-2\Delta \tau} \mathcal{O}_{\text{flat}}(x = e^{-\tau} n) = \frac{1}{x^{2\Delta}} \mathcal{O}_{\text{flat}} \left(\frac{x^\mu}{x^2} \right) \tag{3.40}$$

So the conjugation acts by an inversion. Working out with (3.40) and using conformal algebra one can easily find:

$$D^\dagger = D, \qquad M_{\mu\nu}^\dagger = -M_{\mu\nu}, \qquad K_\mu = P_\mu^\dagger \tag{3.41}$$

Demanding the norm to be positive we take:

$$(P_\mu |\mathcal{O}_I\rangle)^\dagger P_\nu |\mathcal{O}_J\rangle \geq 0 \tag{3.42}$$

using (3.41) and the conformal algebra, one can show that

$$(\Delta \delta_{\mu\nu} - S_{\mu\nu}) \langle \mathcal{O}_I | \mathcal{O}_J \rangle \geq 0 \tag{3.43}$$

To preserve unitarity, $\langle \mathcal{O}_I | \mathcal{O}_J \rangle$ is assumed to be a positive definite matrix. Therefore, the scaling dimension must be greater or equal to the maximum eigenvalues of $S_{\mu\nu}$. It turns out, for each spin l there is a minimal allowed dimension that is consistent with unitarity:

$$\Delta(l) \geq d + l - 2 \quad (l \geq 1), \qquad \Delta(0) \geq (d-2)/2 \quad (l = 0) \tag{3.44}$$

these are called *unitarity bounds*. This is a lower bound on Δ and we will see that bootstrap technology offers an extra, upper bound.
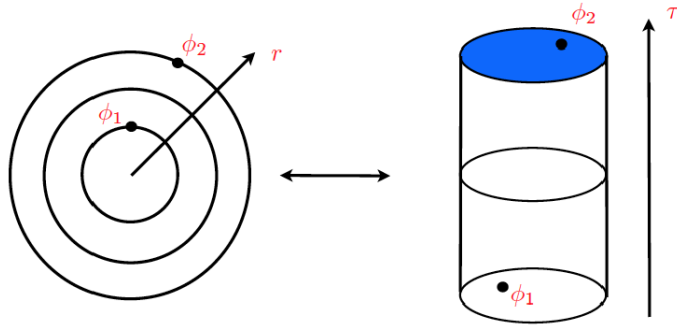
3.6 Operator Product Expansion (OPE)

In QFT, it is expected that if we take a bilocal product of operators, one defined at x and one at y, trying to bring the two points closer together, we can replace this product with a sum of local operators. Inserting two primary operators $\mathcal{O}_i(x)$ and $\mathcal{O}_j(0)$ inside a sphere, will produce a state $|\psi(x)\rangle$ on the sphere. From quantum mechanics, a state $|\psi(x)\rangle$ in the Hilbert space (in radial quantization) have an expansion in terms of the basis $|\Delta_n\rangle$:

$$|\psi(x)\rangle = \sum_n c_n(x)|\Delta_n\rangle \tag{3.45}$$

and since we have inserted an x dependent operator, the states $|\Delta_n\rangle$ contains both the primaries and the descendants. From state-operator correspondence, we have:

$$\mathcal{O}_i(x)\mathcal{O}_j(0)|0\rangle = \sum_{\text{k=primary}} \lambda_{ijk} \, P_k\left(|x-y|, \partial_y\right) \mathcal{O}_k(y)|0\rangle\big|_{y=0} \tag{3.46}$$

where we are summing over primaries provided that a certain polynomial differential operator $P_k(|x-y|, \partial_y)$ is acting on the vacuum and gives us the descendants in addition to the primaries. λ_{ijk} for a unitary theory, are real constant coefficients. A diagrammatic representation of this equation for two scalars is fig.(5) part (a). Expanding the product

$$\mathcal{O}_i(x)\mathcal{O}_j(0) \sim \frac{c}{|x|^{k_1}}[\mathcal{O}(0) + \alpha x^\mu \partial_\mu \mathcal{O}(0) + \dots] + (\text{other primaries + decendents}) \tag{3.47}$$

the first term on the right-hand side is the first primary with its descendants. Acting by D on both sides and using conformal algebra determine k_1 as:

$$k_1 = \Delta_1 + \Delta_2 - \Delta_{k_1} \tag{3.48}$$

Now acting on both sides of (3.47) by K_μ fixes the coefiecent α:

$$\alpha = \frac{\Delta_1 - \Delta_2 + \Delta_{k_1}}{2\Delta_{k_1}} \tag{3.49}$$

and in that way, one can fix all the coefficients appearing for descendants. Hence, the differential operator $P_k(|x-y|, \partial_y)$ is fixed by conformal symmetry. We can rewrite (3.47):

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k \frac{\lambda_{ijk}}{|x|^{\Delta_i+\Delta_j-\Delta_k}} \left[\mathcal{O}_k(0) + \alpha x^\mu \partial_\mu \mathcal{O}_k(0) + \beta x^\mu x^\nu \partial_\mu \partial_\nu \mathcal{O}_k(0) + \dots\right] \tag{3.50}$$

Where λ_{ijk} are constant numbers. This series has a finite radius of convergence and it is known as operator product expansion (OPE). The final form of this product would be

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_k \lambda_{12k} \, P(x_{12}, \partial_2) \, \mathcal{O}_k(x_2) \tag{3.51}$$

the right-hand side is sometimes called the *conformal partial wave*. Any higher-point correlation functions in CFT can be decomposed to two-point functions and that is by using the multiplication of this OPE successively. Therefore, we always get a multiplication of the written right-hand-side terms in any correlation.

4 Conformal Bootstrapping

4.1 Conformal Blocks

Let us use the OPE of two operators to compute a four-point function of identical scalars. We assume x_1 and x_2 are close enough to be surrounded by a sphere. The same is applied for points x_3 and x_4. In that sense, similar to Wick's theorem, we are applying the OPE between the operators $\phi(x_1)$ and $\phi(x_2)$ and separately between $\phi(x_3)$ and $\phi(x_4)$:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \sum_{\mathcal{O}} \lambda^2_{\phi\phi\mathcal{O}} \ P(x_{12}, \partial_2)P(x_{34}, \partial_4) \ \langle \mathcal{O}(x_2)\mathcal{O}(x_4) \rangle \tag{4.1}$$

where we have terminated one sum arguing that the two-point function of the two operators in conformal field theory is diagonal. Exchanging the operators in the left-hand side of (4.1) and relabeling their positions would not change anything but can cause the right-hand side to produce a minus sign if the primary operators on the left are odd spins since they have to be contracted with odd number of space coordinates. Hence, the sum is over even spin primaries. The pictorial representation of this came in fig.(5) part (b).
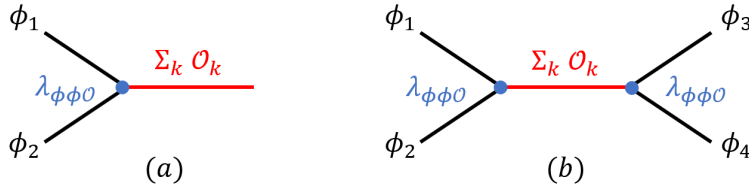

Figure 5: (a) A diagramatic represntation of equ.(3.46). (b) The diagram of four-point function (4.1).

It was shown around 2004 that one can re-sum the contributions of all decedents associated with the primaries and (4.1) takes the form:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{g(u,v)}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}} = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}} \sum_{\mathcal{O}} \lambda^2_{\phi\phi\mathcal{O}} \ g_{\Delta, l}(u,v) \tag{4.2}$$

where $g_{\Delta, l}(u, v)$ for a given scaling dimension and spin, is the function of cross-rations (3.35) and it is called the *global conformal block*. In CFT_2 it is called *Virasoro conformal block*. This function is invariant under conformal transformations. The expression (4.2) is a general argument. One can find out the structure of conformal blocks by studying the correlation of primaries explicitly. The two-point function of spin-l primaries is:

$$\langle \mathcal{O}^{\mu_1 \dots \mu_l}(x_1) \mathcal{O}_{\nu_1 \dots \nu_l}(x_2) \rangle = \frac{I^{(\mu_1}_{(\nu_1} \dots I^{\mu_l)}_{\nu_l)} - (\text{traces})}{(x_{12})^{2\Delta}}, \qquad I^\mu_\nu = \delta^\mu_\nu - \frac{2x^\mu x_\nu}{x^2} \tag{4.3}$$

with l Lorentz indicies. Putting this into (4.1), we can get:

$$g_{\Delta, l}(u,v) = \left(x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi} \right) \frac{P_{\mu_1 \dots \mu_l}(x_{12}, \partial_2) \ P_{\nu_1 \dots \nu_l}(x_{34}, \partial_4) \ I^{\mu_1 \dots \mu_l \ \nu_1 \dots \nu_l}(x_{24})}{\lambda^2_{\phi\phi\mathcal{O}} \ x_{24}^{2\Delta_\phi}} \tag{4.4}$$

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In the limit, $x \to 0$ ($x_1 \to x_2$, $x_3 \to x_4$) the asymptotic expression for P is:

$$P_{\mu_1 \ldots \mu_l}(x, \partial) \sim \frac{x_{\mu_1} \ldots x_{\mu_l}}{x^{2\Delta_\phi - \Delta + l}} \tag{4.5}$$

This limit is equivalent to sending $u \to 0$ and $v \to 1$ which leads to an approximate value of (4.4):

$$g_{\Delta, l}(u, v) \sim (-2)^{-l} \, u^{(\Delta - l)/2} \, (1 - v)^l \tag{4.6}$$

One way to find an explicit expression for conformal blocks is to use an appropriate differential operator acting on the block and then solve the resulting differential equation. We are seeking the operator that acts on each state in the multiplet[4] with the same eigenvalue. Those operators are called *Casimir* operators and we want to find the quadratic Casimir of the conformal group since it encodes the maximal algebraic structure. It would be the combination of the conformal generators. As discussed before, In Euclidean signature, the conformal group is isomorphic with $SO(d + 1, 1)$. We can define a matrix that is the generator of Lorentz transformations in this $d + 2$ dimensional space, simultaneously made of a certain combination of conformal generators:

$$\mathcal{M}_{AB} = \begin{pmatrix} M_{\mu\nu} & -\frac{1}{2}(P_\mu + K_\mu) & \frac{1}{2}(P_\mu - K_\mu) \\ \frac{1}{2}(P_\mu + K_\mu) & 0 & -D \\ -\frac{1}{2}(P_\mu - K_\mu) & D & 0 \end{pmatrix} \tag{4.7}$$

where the indices A and B run from 0 to $d + 1$. This satisfies the Lorentz algebra:

$$[\mathcal{M}_{AB}, \mathcal{M}_{CD}] = i(\eta_{AC}\mathcal{M}_{BD} + \eta_{BD}\mathcal{M}_{AC} - \eta_{AD}\mathcal{M}_{BC} - \eta_{BC}\mathcal{M}_{AD}) \tag{4.8}$$

Hence the Casimir operator takes the form

$$\mathcal{C} = \frac{1}{2}\mathcal{M}_{AB}\mathcal{M}^{AB} = \frac{1}{2}M_{\mu\nu}M^{\mu\nu} + P_\mu K^\mu + D(D - d) \tag{4.9}$$

one can explicitly write the differential representation for each generator and get the differential form of Casimir in terms of cross-ratios non-trivially as

$$\mathcal{D} = (1 - u - v)\partial_u u \partial_v + u \partial_u (2u\partial_u - d) + (1 - u - v)(u\partial_u + v\partial_v)(u\partial_u + v\partial_v) \tag{4.10}$$

this acts on the multiplet and gives the same eigenvalue for states therein. For every given primary, the conformal block basically takes care of all the contributions of the descendants, so the Casimir will act on the conformal block itself with a unique eigenvalue:

$$\mathcal{D} \, g_{\Delta, l}(u, v) = c_{\Delta, l} \, g_{\Delta, l}(u, v), \qquad c_{\Delta, l} = \Delta(\Delta - d) + l(l + d - 2) \tag{4.11}$$

In order to solve this second-order differential equation it is useful to change the variables:

$$u = z\bar{z}, \qquad v = (1 - z)(1 - \bar{z}) \tag{4.12}$$

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[4]The multiplet here is the operator along with all its descendants which are representations in the conformal group.

in terms of complex variables z and $\overline{z}$, (4.10) takes the form:

$$\mathcal{D} = z(z^2(1-z)\partial_z^2 - z^2\partial_z) + z(\overline{z}^2(1-\overline{z})\partial_{\overline{z}}^2 - \overline{z}^2\partial_{\overline{z}}) \tag{4.13}$$

$$+ 2(d-2)\frac{z\overline{z}}{z-\overline{z}}((1-z)\partial_z - (1-\overline{z})\partial_{\overline{z}})$$

and therefore the solution to the differential equation (4.11) in 2 and 4 dimensions is:

$$g_{\Delta,l}(z,\overline{z}) = k_{\Delta+l}(z)k_{\Delta-l}(\overline{z}) + k_{\Delta-l}(z)k_{\Delta+l}(\overline{z}), \qquad \text{for d=2} \tag{4.14}$$

$$g_{\Delta,l}(z,\overline{z}) = \frac{z\overline{z}}{z-\overline{z}}(k_{\Delta+l}(z)k_{\Delta-l-2}(\overline{z}) - k_{\Delta-l-2}(z)k_{\Delta+l}(\overline{z})), \qquad \text{for d=4} \tag{4.15}$$

$$k_\beta(x) = x^{\beta/2} \,_2F_1\left(\frac{\beta}{2}, \frac{\beta}{2}; \beta|x\right) \tag{4.16}$$

where $_2F_1$ is the hypergeometric function of z and $\overline{z}$. In odd dimensions, there is no solution to the differential equation in terms of elementary functions z and $\overline{z}$.

Changing variables u, v to $z, \overline{z}$ would put the points x_1 to x_4 on the complex plane. By special conformal and translation transformation, we can send x_4 to ∞ and x to 0. We can do rotation and put x_3 on the same axis on an arbitrary point, for example at $x_3 = 1$. Since we used the three conformal transformations, generally the point x_2 can be located at an absolute value of z. In radial coordinates $\{r, \theta\}$ with the complex variable $\rho = re^{i\theta}$, it is possible to translate the points on a complex plane and put them on circles (fig.(7)).
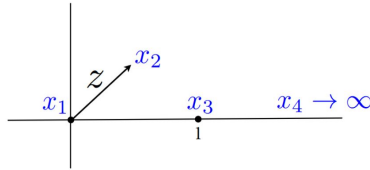

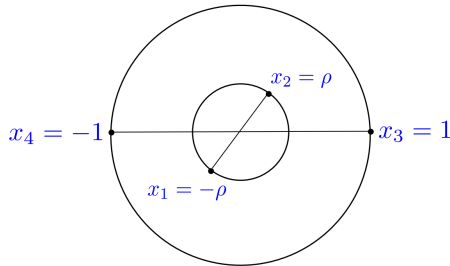Figure 6: Conformal frame defining the z coordinate. Figure taken from [9].

Figure 7: Conformal frame defining the radial coordinate. Figure taken from [9].

The three constants λ_{ijk}, Δ, and l are collectively the *CFT data*, and they differ from one CFT to another. One can not determine these unique constants using conformal symmetry. In any higher-point function, we might look at, we can successively use the OPE to reduce

it down to a two-point function. The only undetermined things that come in the final result would be the three-point function coefficients λ_{ijk}, Δ, and l. In principle, Knowing this set would allow us to determine any correlation function in a conformal field theory. The procedure which will allow us to extract information about these three constants, from four-point function decomposition in terms of conformal blocks is the crossing symmetry which is going to be discussed in the next section.

4.2   Crossing Symmetry

To analyze the four-point correlation function, we can fix three points (x_1, x_3, x_4) and vary the position of the point x_2. As we move x_2 across the complex plane, we can consider two distinct ways in which the operators can be ordered: the s-channel (direct channel) and the t-channel (crossed channel). Conformal invariance dictates that the two channels are equal by crossing symmetry:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle \tag{4.17}$$

The final answer would only be the exchange of the positions of the operators in space which leaves the answer invariant.

Where we can take the final result in terms of 2-point function convergent OPEs. It has been proven that the two channels' decompositions match. This is illustrated in Fig. (8). Now for four identical scalars we have

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \tag{4.18}$$

Similar to (4.2), if we now take the OPE decompositions $\phi(x_1)\phi(x_4)$ and $\phi(x_2)\phi(x_3)$ in the four-point functions, that yields

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(v,u)}{(x_{14}^2 x_{23}^2)^{\Delta_\phi}} = \frac{1}{(x_{14}^2 x_{23}^2)^{\Delta_\phi}} \sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}}^2 \; g_{\Delta,l}(v,u) \tag{4.19}$$

the expressions (4.2) and (4.19) must be equal imposing the crossing symmetry condition. Equating these two would give

$$\sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}}^2 \left(g_{\Delta,l}(u,v) - \left(\frac{u}{v}\right)^{\Delta_\phi} g_{\Delta,l}(v,u) \right) = 0 \tag{4.20}$$

this is called *crossing equation* and it is the starting point of conformal bootstrap.

In the next section, we use equation (4.20) to introduce numerical bootstrap technics and to demonstrate the resulting behavior of conformal theories in various dimensions. We will put constraints on this equation, to only get the unitary systems which will eventually confine the values of CFT data i.e. Δ, λ_{ijk} and l. We will see that some analysis of conformal bootstraps agrees perfectly with the data we have from some original quantum field theories which are significantly interesting.
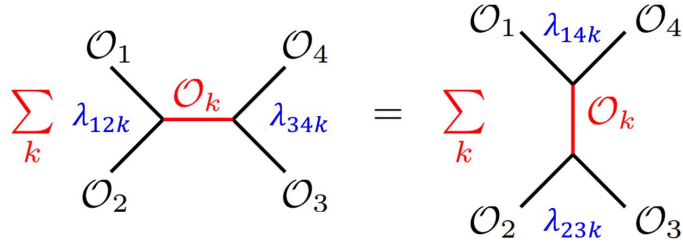
Figure 8: Crossing symmetry condition that leads to crossing equation (4.20) for four identical scalars.

4.3 Numerical Bootstrap

Unlike analytical bootstrap, Numerical bootstrap is a powerful way to study CFTs and extract CFT data such as information about unitarity, scaling dimension, and spins. This is also a useful way to match experimental universal data of the known theories with data coming from scale-invariant theories at fixed points. This data also provide a nonperturbative framework to study theories that exist in the low-energy regime. Although the examples of this method that will be discussed in this section, are not understood analytically, it aims at a bright direction for future works.

It is common in the bootstrap literature to take the parentheses in (4.20) and denote it by $\mathcal{F}_{\Delta,l}(u,v)$ which is sometimes called the convolved conformal block:

$$\sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}}^2 \, \mathcal{F}_{\Delta,l}(u,v) = 0 \tag{4.21}$$

We are going to consider a four-dimensional case of a free scalar field with $\Delta_\phi = 1$ saturating the unitarity bound. Working in $z\bar{z}$ coordinates with the choice $z = \bar{z}$ allows us to plot $F_{\Delta,l}(z = \bar{z})$ on a plane rather than in three dimensions. To plot F in terms of z, different choices of free parameters Δ and l have been made (see fig.(9)).
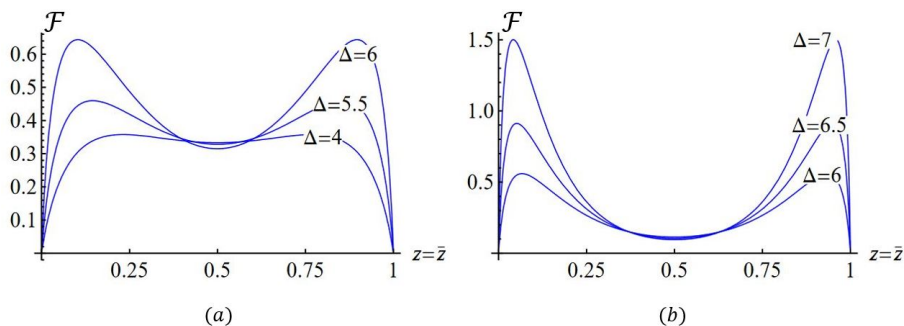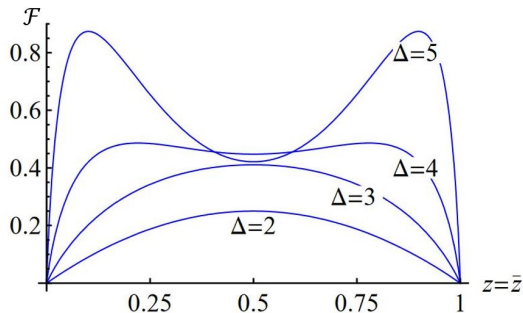

Figure 9: Convolved conformal block of four identical scalars is drawn for different values of z and Δ. For (a) $l = 2$ and (b) $l = 4$. Both diagrams have a local minimum at $z = 0.5$ which approves (4.22) [10].

In this plot, one finds that:

$$F_{\Delta,l}'' > 0 \tag{4.22}$$

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In a unitary theory $\lambda_{\phi\phi\mathcal{O}}$ hence positive. So taking the second derivative of $\mathcal{F}$, would obviously be greater than zero. This tells us that the scalar field OPE with itself must contain a scalar too. For $l = 0$ we have a similar plot that at small Δ and $z = 1/2$ contains a negative second derivative of $\mathcal{F}_{\Delta,l}$ (see fig.(10)). This also has an upper bound for these scalars in which at a critical value of scaling dimension Δ_c, we would again get a positive second derivative. Therefore, in $\phi \times \phi$ OPE there needs to exist at least one scalar with $\Delta \leq \Delta_c$.

Figure 10: Convolved conformal block of four identical scalars is drawn for different values of z and Δ but now for $l = 0$. It includes a maximum at $z = 0.5$ which result negative second-derivative [10].

- Constraints on OPE coefficients

We take out two operators out of the sum in (4.21): an arbitrary operator $\mathcal{O}_0$ with dimension Δ_0 and spin l_0 and the identity with $\Delta = 0$ and $l = 0$:

$$\lambda_{\phi\phi\mathcal{O}_0}^2 \ \mathcal{F}_{\Delta_0,l_0} = -\mathcal{F}_{0,0} - \sum_{\mathcal{O}\neq 1, \mathcal{O}_0} \lambda_{\phi\phi\mathcal{O}}^2 \ \mathcal{F}_{\Delta,l} \tag{4.23}$$

where OPE coefficient for identity is one. One can introduce a functional $\alpha(\mathcal{F})$ that takes $\mathcal{F}$ as an entry and give a real number as an outcome, assuming $\alpha(\mathcal{F}) \geq 0$. We are able to assume that for $\mathcal{O}_0$ this functional has an outcome value of one. Indeed we can always choose the renormalization in a way that this will happen. In that case:

$$\lambda_{\phi\phi\mathcal{O}_0}^2 = -\alpha(\mathcal{F}_{0,0}) - \sum_{\mathcal{O}\neq 1, \mathcal{O}_0} \lambda_{\phi\phi\mathcal{O}}^2 \ \alpha(\mathcal{F}_{\Delta,l}) \tag{4.24}$$

the summation in the right-hand side is always positive, therefore:

$$\lambda_{\phi\phi\mathcal{O}_0}^2 \leq -\alpha(\mathcal{F}_{0,0}) \tag{4.25}$$

which naturally shows that there is an upper bound for OPE coefficients. One can take this functional to be of the form

$$\alpha = \sum_{m+n\leq\Lambda} \alpha_{mn} \partial_z^m \partial_{\bar{z}}^n \mathcal{F}_{\Delta,l}(z,\bar{z})|_{z=\bar{z}=1/2} \ \geq 0 \tag{4.26}$$

where α_{mn} are constants that have to be determined numerically. A second derivative form of this function appears since we are interested in the behavior (4.22). The reason

that one is interested in evaluating this functional on a certain point $z = \bar{z} = 1/2$ is that both terms of (4.20) will agree at this point and it turns out it plays an important role in the convergence of OPE in higher dimensions.

  • Constraints on operators scaling dimension

Starting from (4.21), one can only pull the identity operator out of the sum and write this expression in terms of specific functional α:

$$-\alpha(\mathcal{F}_{0,0}) = \sum_{\mathcal{O} \neq 1} \lambda^2_{\phi\phi\mathcal{O}_0} \alpha(\mathcal{F}_{\Delta,l}) \tag{4.27}$$

We demand that the functional be something such that: $\alpha(\mathcal{F}_{0,0}) = 1$ and $\alpha(\mathcal{F}_{\Delta,l}) \geq 0$ with these assumptions:

$$-1 = \sum_{\mathcal{O} \neq 1} \lambda^2_{\phi\phi\mathcal{O}} \; \alpha(\mathcal{F}_{\Delta,l}) \tag{4.28}$$

which is inconsistent with unitarity since the right-hand side is positive. One can find another functional with which these conditions are satisfied in addition to excluding the corresponding spectrum. By increasing Λ in (4.26) to more derivatives, we get access to higher values of scaling dimension and it is possible to find such a function. It has been argued by numerical calculation that checking the higher derivatives would change the corresponding dimension of the operator. This can happen, by imposing the constraint that the leading spin zero operator has a delta larger than some value. We can denote the leading operator dimension as Δ_{ϕ^2} since we have in mind that the leading nonperturbative contribution after the identity would come from two spin-zero operators in $\phi \times \phi$ OPE. Requiring a functional α to exist, would give us two regions in different values of dimensions. Allowed and disallowed values (see fig. (11)). It is also constrained that unitarity bonds (3.44) exist and the λ coefficients are positive.
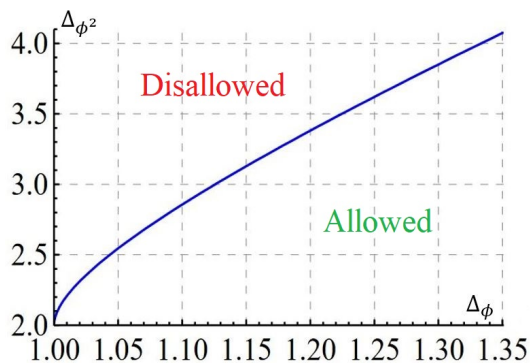

Figure 11: Bound on Δ values for scalar field OPE in $d = 4$. The same analysis applies to other dimensions and fields [10].

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- Wilson-Fisher fixed point

Since the IR and UV fixed points are universal conformally invariant points in the space of theories, getting solutions at critical points such as critical exponent must be possible and independent of studying the whole RG flow of the theories. We will shortly see that one can find such a fixed point by bootstrap in three dimensions. Consider the following scalar theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi \ \partial^\mu\phi - \frac{1}{4!}\lambda\phi^4 \tag{4.29}$$

This theory possesses $\mathbb{Z}_2$ symmetry. The dimension of ϕ^4, $\Delta_{\phi^4} = 2(d-2)$ shows that this theory has a strongly relevant coupling which means it does not have a perturbative flow to analyze the theory. One way to deal with this is to use ϵ-expansion $d = 4 - \epsilon$ and perform perturbative series over ϵ which will produce RG flow. This method is sometimes referred to as Wilson-Fisher formalism. This theory also includes a mass term which we have ignored since we are interested in studying the fixed points. The theory where $\epsilon \to 1$ has the same universality class as the Ising model in $d = 3$. This is a perturbative approach. However, one can study this theory non-perturbatively by Bootstrap. The computation is similar to what has been discussed and the result is given by fig.(12).
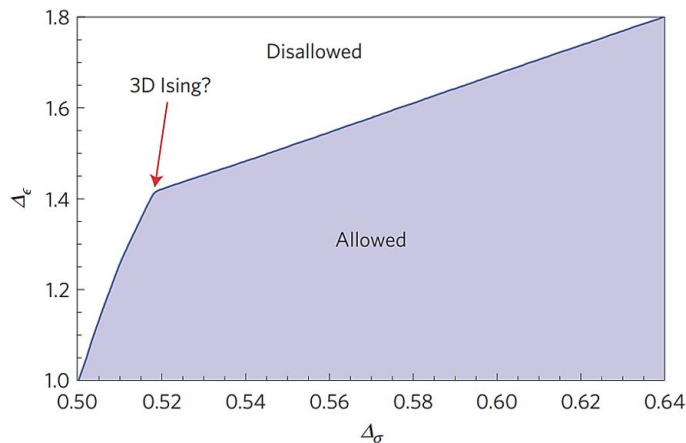

Figure 12: Upper bound on Δ_ϵ as a function of Δ_σ in 3d CFTs [14]. The kink represents the 3d critical Ising model, the same as Wilosn-Fisher fixed point. Here ϕ and ϕ^2 are denoted by σ and ϵ respectively.

In the CFT context, a $\mathbb{Z}_2$ symmetry imposes selection rules on the possible operators appearing in different OPE channels. Let us take a $\mathbb{Z}_2$-odd scalar operator σ and consider the $\sigma \times \sigma$ OPE. It can only contain $\mathbb{Z}_2$-even operators: $\sigma \times \sigma \sim 1 + \lambda_{\sigma\sigma\epsilon}\epsilon + \lambda_{\sigma\sigma T}T^{\mu\nu} + \dots$. Here, 1 is the identity operator, ϵ is the leading $\mathbb{Z}_2$-even scalar, $T_{\mu\nu}$ is the stress-energy tensor, and so on. In this setup, we would like to ask what is the maximal allowed value of Δ. A numerical bootstrap analysis of the 4pt function $\langle\sigma\sigma\sigma\sigma\rangle$ produces an upper bound on Δ_ϵ as a function of Δ_σ, shown in fig.(12). The point $\{1/2, 1\}$ corresponds to the theory of a free massless scalar while the point $\{0.518, 1.413\}$, sitting near a discontinuity in the boundary, corresponds to the critical 3d Ising model. Other theories that live in the interior of this region are the critical $O(N)$ models.

- Mixed correlator bootstrap

One can ask what is the effect of adding constraints from other 4pt functions. So far the main system that has been studied in the literature is $\langle\sigma\sigma\sigma\sigma\rangle, \langle\sigma\sigma\epsilon\epsilon\rangle, \langle\epsilon\epsilon\epsilon\epsilon\rangle$, though other systems may also prove interesting. An advantage of including the mixed correlator $\langle\sigma\sigma\epsilon\epsilon\rangle$ is that it allows one to probe the $\mathbb{Z}_2$-odd operators appearing in the OPE:

$$\sigma \times \epsilon \sim \lambda_{\sigma\epsilon\sigma}\sigma + \lambda_{\sigma\epsilon\sigma'}\sigma' + \ldots \tag{4.30}$$

Instead of just imposing crossing symmetry constraint on the four-point function of identical scalars ϕ, one can also apply it to the mixed correlations $\langle\sigma\sigma\epsilon\epsilon\rangle$. Before we only had access to the spectrum of even spin operators but now we are able to read the information from odd spins too. This would give us stronger constraints. To obtain crossing equations, one has to work out different decompositions of and $\langle\sigma\sigma\sigma\sigma\rangle, \langle\sigma\sigma\epsilon\epsilon\rangle$, and $\langle\epsilon\epsilon\epsilon\epsilon\rangle$ impose crossing symmetry constraint. In this case, one would have five crossing equations that take the form:

$$\mathcal{F}^{ij,kl}_{\pm,\Delta,l} = v^{\frac{\Delta_j+\Delta_k}{2}} g^{(\Delta_{ij},\Delta_{kl})}_{\Delta,l}(u,v) \pm u^{\frac{\Delta_j+\Delta_k}{2}} g^{(\Delta_{ij},\Delta_{kl})}_{\Delta,l}(v,u), \tag{4.31}$$

where $+$ and $-$ refer to even and odd spin contributions respectively. The idicies i,j,k,l are representations of operators σ and ϵ. Now the conformal blocks are more complicated and depend also on the difference of dimensions: $\Delta_{ij} = \Delta_i - \Delta_j$. We can rewrite (4.31) in vectorial representation:

$$\sum_{\mathcal{O}^+} (\lambda_{\sigma\sigma\mathcal{O}} \quad \lambda_{\epsilon\epsilon\mathcal{O}})\vec{V}_{+,\Delta,l}\begin{pmatrix}\lambda_{\sigma\sigma\mathcal{O}}\\ \lambda_{\epsilon\epsilon\mathcal{O}}\end{pmatrix} + \sum_{\mathcal{O}^-} \lambda^2_{\sigma\epsilon\mathcal{O}}\vec{V}_{-,\Delta,l} = 0, \tag{4.32}$$

with two vectors

$$\vec{V}_{-,\Delta,l} = \begin{pmatrix} 0 \\ 0 \\ \mathcal{F}^{\sigma\epsilon,\sigma\epsilon}_{-,\Delta,l}(z,\overline{z}) \\ (-1)^l \mathcal{F}^{\epsilon\sigma,\sigma\epsilon}_{-,\Delta,l}(z,\overline{z}) \\ -(-1)^l \mathcal{F}^{\epsilon\sigma,\sigma\epsilon}_{+,\Delta,l}(z,\overline{z}) \end{pmatrix}, \quad \vec{V}_{+,\Delta,l} = \begin{pmatrix} \begin{pmatrix} \mathcal{F}^{\sigma\sigma,\sigma\sigma}_{-,\Delta,l}(z,\overline{z}) & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{F}^{\epsilon\epsilon,\epsilon\epsilon}_{-,\Delta,l}(z,\overline{z}) \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{2}\mathcal{F}^{\sigma\sigma,\epsilon\epsilon}_{-,\Delta,l}(z,\overline{z}) \\ \frac{1}{2}\mathcal{F}^{\sigma\sigma,\epsilon\epsilon}_{-,\Delta,l}(z,\overline{z}) & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{2}\mathcal{F}^{\sigma\sigma,\epsilon\epsilon}_{+,\Delta,l}(z,\overline{z}) \\ \frac{1}{2}\mathcal{F}^{\sigma\sigma,\epsilon\epsilon}_{+,\Delta,l}(z,\overline{z}) & 0 \end{pmatrix} \end{pmatrix} \tag{4.33}$$

Now one can apply the same technique as before, using a vector functional $\vec{\alpha} = (\alpha_1, \ldots \alpha_5)$ and take the inner product of this functional with each crossing equation of (4.32) to check unitarity upper bound for various dimensions. Each equation will have an expansion in the derivatives and α_{mn} coefficients.

In [15] it was found that with no assumptions this system leads to an allowed region identical to Fig. (12), while by inputing the assumption of a single relevant $\mathbb{Z}_2$-odd operator (i.e., $\Delta_{\sigma'} > 3$) it leads to the allowed region shown in Fig. (13). In this plot, one can see a detached *island* containing the critical Ising model as well as a *bulk* region further to the right. This "bulk" region has so far not been systematically explored in the literature: it would be very interesting to understand what other CFTs lie inside of it.
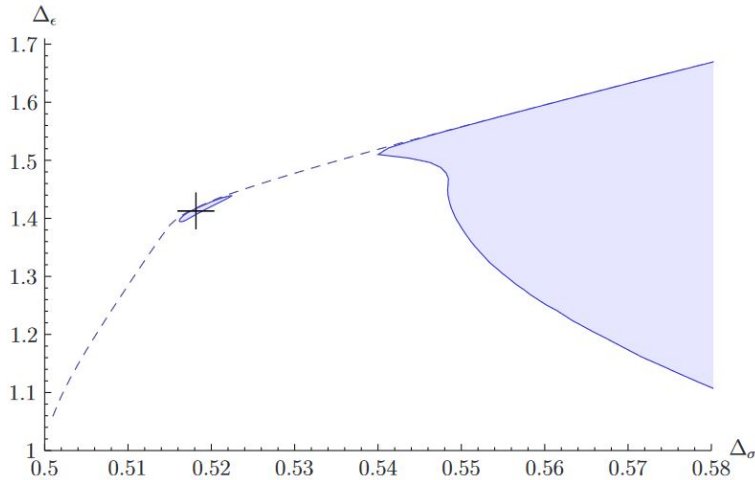
Figure 13: Allowed region following from the analysis of three 4pt functions assuming $\Delta_{\sigma'} \geq 3$ with no assumption on $\Delta_{\epsilon'}$ [15].

- Global Symmetry: 3D O(N) Models

Four-point correlation functions of CFT operators that are in irreducible representations of the global symmetry G can be organized using group theory and decomposed into different G-invariant tensor structures. We have N scalar fields and we put them all in the victor representation of $O(N)$ symmetry. The first numerical analyses of the resulting equations occurred in the context of 4d CFTs [11, 12, 13], but the group-theoretic structure is d-independent. The bootstrap for $O(N)$ symmetry in 3d was investigated by [17], [18] and [19]. We will start our analysis assuming that the CFT contains an operator $\phi \equiv (\phi_a)_{a=1}^N$ in the fundamental representation of $O(N)$, of dimension Δ_ϕ. We would like to learn about the operators in the OPE $\phi_a \times \phi_b$. By group theory, operators of even spin l in this OPE will transform as $O(N)$ singlets S or symmetric traceless tensors of rank-2, T^{ab}, while odd-spin operators will transform in the rank-2 antisymmetric representation $A^{[ab]}$:

$$\phi^a \times \phi^b \sim \delta^{ab} S + T^{(ab)} + A^{[ab]} \tag{4.34}$$

The four-point function for these victors is given by:

$$\langle \phi^a \phi^b \phi^c \phi^d \rangle = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}} \sum_I \sum_{\mathcal{O}_I} \mathcal{P}_I^{abcd} \lambda_{\phi\phi\mathcal{O}}^2 \; g_{\Delta_I, l_I}(u, v) \tag{4.35}$$

where $\mathcal{P}$ is some projector that tells us what the two-point function of S, $T^{(ab)}$ and $A^{[ab]}$ looks like. The projector for each of these at the level of the 4-point function is:

$$
\begin{aligned}
S: & \quad \mathcal{P}_1 \sim \delta^{ab} \delta^{cd} \\
T^{(ab)}: & \quad \mathcal{P}_2 \sim \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} - \frac{2}{N} \delta^{ab} \delta^{cd} \\
A^{[ab]}: & \quad \mathcal{P}_3 \sim \delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc}
\end{aligned}
\tag{4.36}
$$

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They are independent projectors, and they decompose the crossing equation into three separate equations. For every four indexes of each projector, we get one crossing equation. Writing different decompositions of the four-point function (4.35), will relate operators in different representations. We again can take a vector functional α with three entries, multiply it with crossing equations, and extract numerical data for the operator dimension bounds. From the crossing relations for the 4pt function of ϕ one can put upper bounds on the dimensions of various operators.

For the lowest dimension scalars ($l = 0$) in the singlet (S) and symmetric traceless tensor (T) sector, these bounds are shown in Fig. (14) and (15) as a function of Δ_ϕ for various values of N.
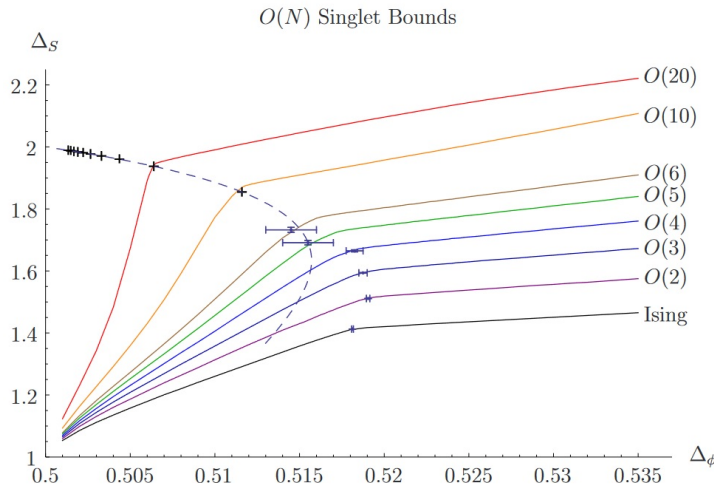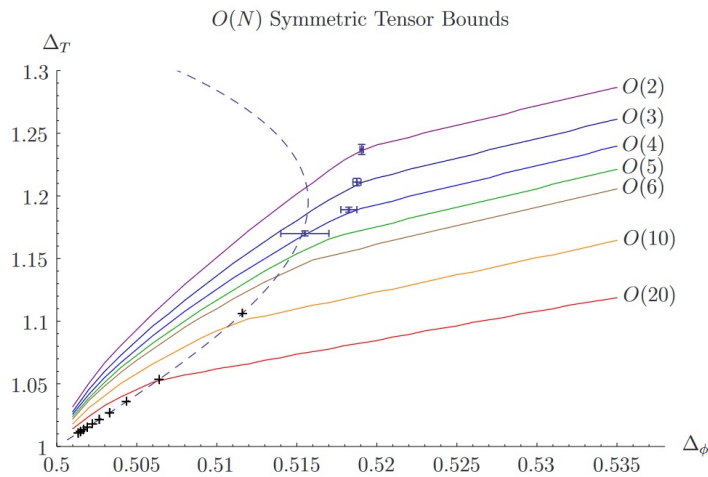

Figure 14: Upper bounds on the dimension of s [16].

Figure 15: Upper bounds on the dimension of t [16].

5 Discussion and Remarks

The setup discussed in this paper has various applications in AdS/CFT correspondence. For example one can ask: What object in AdS computes a conformal block? The answer is a *Witten diagram*. A Witten diagram is a geometrical method to calculate the CFT correlation functions using bulk information. They are analogous to Feynman diagrams in quantum field theory, but instead of integrating over internal lines and vertices, they integrate over bulk fields and interactions in AdS. Witten diagrams can be computed using the AdS/CFT dictionary, which relates bulk fields to boundary operators and bulk interactions to boundary OPE coefficients. These diagrams can be decomposed into conformal blocks, which are universal functions that encode the contribution of a primary operator and its descendants to the correlation function. Conformal blocks depend only on the conformal dimensions and spins of the operators involved. For example, according to [20], the conformal blocks for one-point functions on the torus can be written as Witten diagrams in thermal AdS. The explicit formula for these conformal blocks is given in [21], which involves a sum over bulk-to-boundary propagators and a conformal Casimir operator. In the holographic dual, the four-point function can be computed by a Witten diagram that consists of four external lines corresponding to the boundary operators $\mathcal{O}_i$ with dimensions Δ_i, connected by a bulk-to-bulk propagator corresponding to an exchanged operator ϕ_Δ^l in AdS (see fig. (16)). The Witten diagram is given by

$$W = \int d^3x \; d^3y \; \sqrt{-g} \; K_{\Delta_1}(x, x_1) K_{\Delta_2}(x, x_2) G_\Delta(x, y) K_{\Delta_3}(y, x_3) K_{\Delta_4}(y, x_4) \tag{5.1}$$

where $K_{\Delta_i}(x, x_i)$ are bulk-to-boundary propagators that relate bulk fields with boundary operators, $G_\Delta(x, y)$ is a bulk-to-bulk propagator that propagates a bulk field between two points in AdS, x, y which are the position of the vertices and the integration variables in AdS. The Witten diagram can be shown to reproduce the same OPE expansion as the CFT four-point function. Therefore, it provides a geometric way of computing conformal blocks in AdS/CFT.
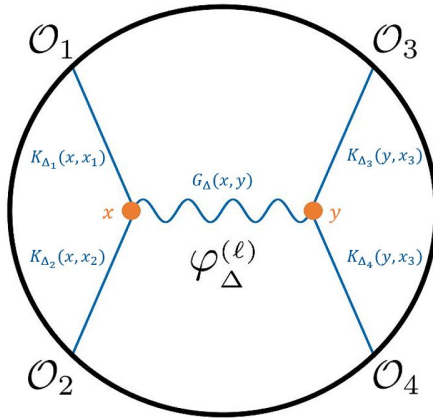

Figure 16: Tree-level Witten diagram for four boundary operators. ϕ_Δ^l is a symmetric traceless spin-l tensor field of dual conformal dimension Δ. Orange dots denote vertices integrated over all of AdS.

Conformal bootstrap is a method for CFTs using only their basic properties such as unitarity, locality, and symmetry. CFTs are important for describing critical phenomena, quantum field theories, and quantum gravity [2]. One of the applications of conformal bootstrap is to constrain and solve CFTs that are dual to string theory or M-theory in anti-de Sitter (AdS) space via the AdS/CFT correspondence [22]. This can help us understand the low-energy effective action of string theory/M-theory and test its consistency [22]. Another application is to determine the properties of specific CFTs in various dimensions, such as the critical exponents and correlation functions of the Ising and O(N) models in three dimensions [2]. This can help us compare theoretical predictions with experimental data and explore new phases of matter.

One can also study correlation functions of the bulk stress tensor and boundary operators in QFTs in AdS space. In particular, one can derive some rules from the two-point function of the stress tensor and its three-point function with two boundary operators. In AdS$_2$, this leads to a bootstrap setup that involves the central charge of the UV limit of the bulk QFT and may allow us to follow an RG flow non-perturbatively by continuously varying the AdS radius [24].

It would further provide us to constrain the spectrum and interactions of higher-spin fields in AdS space by bootstrapping the dual CFTs with higher-spin symmetry. It also can study the entanglement entropy and Renyi entropy of CFTs with boundaries or defects by using the AdS/BCFT correspondence and the holographic formula for entanglement entropy. It is also possible to Explore the phase structure and thermodynamics of black holes in AdS space by bootstrapping the dual CFTs at finite temperature. Testing the consistency of quantum gravity in AdS space by bootstrapping the dual CFTs with large central charge and sparse spectrum, require careful consideration, although it could lead to interesting consequences.

Acknowledgments

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