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Classical Liouville Action and Uniformization of Orbifold Riemann Surfaces: A Geometric Approach to Classical Correlation Functions of Branch Point Vertex Operators

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Background & Motivation

Liouville Theory first appeared as a conformal anomaly in Polyakov's attempt to understand non-critical bosonic strings. Since then, Liouville theory has found various applications in both physics and mathematics.

From the point of view of low-dimensional quantum gravity and holography, there has been a resurgence of interest in Liouville Theory due to its close connection with JT gravity and AdS_3/CFT_2.[Krasnov '00 a&b/Krasnov '01/Krasnov '02/Krasnov, Schlenker / Takhtajan, Teo '06/ Mertens, Turiaci '21]

This theory admits two dimensional surfaces of constant negative curvature (possibly with sources) as its classical solutions: Let X be a compact closed Riemann surface of genus $g > 1$. In the absence of sources, complete conformal metrics $ds^2 = e^{\phi(u,\bar{u})}|du|^2$ on X are classical fields of this theory, and the Liouville equation $-2e^{-\phi}\partial_{\bar{u}}\partial_u\phi = -1$ is the corresponding Euler-Lagrange equation.

According to the uniformization theorem, the hyperbolic metric on X is the unique classical solution of the theory and one can consider this classical solution as the critical point of a certain functional defined on the space of all smooth conformal metrics on X. This functional is called the Liouville action functional and its critical value — the classical Liouville action.

The definition of the classical Liouville action is a non-trivial problem: Since $\phi(u, \bar{u})$ is not a globally defined function on X, but rather a logarithm of the conformal factor of the metric, the "kinetic term" $|\partial_u \phi|^2 du \wedge d\bar{u}$ does not yield a (1,1)-form on X and, therefore, can not be integrated over X.

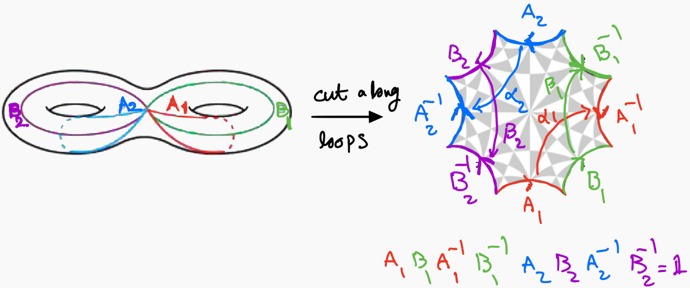
Takhtajan and Zograf solved this problem by using global coordinates, provided by different uniformizations of X: Instead of defining the Liouville action in terms of classical fields on X, one chooses to define this action in terms of Liouville field on a planar covering of X.

Such a planar covering can be found uniquely given a "marking" of X and are in one-to-one correspondence with complex projective structures on X. More precisely, such a geometric structure on X can be viewed as a $(\mathrm{PSL}(2,\mathbb{C}), \mathbb{CP}^1)$-structure defined via an open cover $\{U_a\}_{a \in A}$ of X with holomorphic charts $f_a : U_a \to \mathbb{CP}^1$ such that the transition functions are given by the restrictions of Möbius transformations.

The fact that classical Liouville action depends on complex projective structures is a manifestation of so-called conformal anomaly.

Fuchsian Uniformization

Compact Riemann surfaces admit several different descriptions: By classical uniformization theorem, every hyperbolic Riemann surface X (i.e. $\chi(X) < 0$) can be realized as a quotient of $\mathbb{H} \cong \mathbb{D}$ by a Fuchsian group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{PSU}(1,1)$:



Let $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ be the generators of $\Gamma \cong \pi_1(X)$; A Fuchsian group with a distinguished system of generators will be called marked. These generators correspond to a canonical homotopy basis of X and marked Fuchsian groups $\longleftrightarrow$ Riemann surfaces with homotopy marking.

Schottky Uniformization

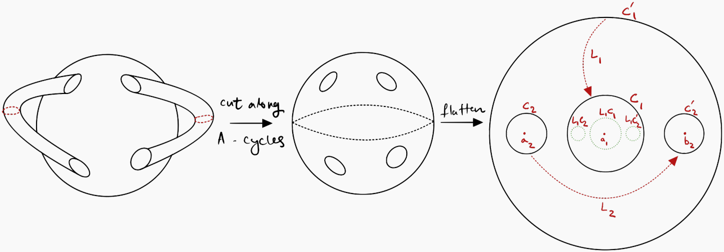
When X is a compact Riemann surface with $g > 1$ handles and no boundaries, the most convenient way to realize X is as a quotient space $\Omega(\Sigma)/\Sigma$. Here, the Schottky group Σ (of rank $g > 1$) is a subgroup of $\mathrm{PSL}(2, \mathbb{C})$ that is freely generated by g loxodromic elements and the region of discontinuity $\Omega(\Sigma)$ is a subregion of $\hat{\mathbb{C}}$ on which Σ acts discontinuously. A Schottky group Σ with a distinguished system of generators $L_1, \ldots, L_g$ will be called marked.

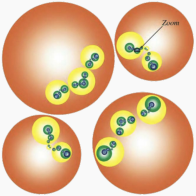
Consider a Marked Fuchsian group Γ and let $\mathcal{N}$ be the smallest normal subgroup in Γ that contains $\alpha_1, \ldots, \alpha_g$. Then, there exists a Schottky group $\Sigma \cong \Gamma/\mathcal{N}$ such that $\mathbb{H}/\Gamma \cong X \cong \Omega/\Sigma$. This Schottky group is marked by generators $L_1, \ldots, L_g$ corresponding to the cosets $\beta_1\mathcal{N}, \ldots, \beta_g\mathcal{N}$.

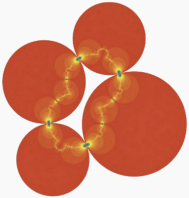
A marked Schottky group is most conveniently described by its fundamental region: It is a subset $\mathcal{D} \subsetneq \Omega$, such that no two distinct interior points of $\mathcal{D}$ are Σ-equivalent, and every point of Ω is Σ-equivalent to some point of $\mathcal{D}$. More specifically, $\mathcal{D}$ can be viewed as the exterior of $2g$ non-intersecting circles $C_1, \ldots, C_g, C_1', \ldots, C_g'$ in $\hat{\mathbb{C}}$, such that $C_i' = -L_i(C_i)$ and the region exterior to C_i is mapped to the interior of C_i'.

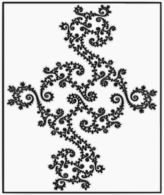
Schottky Uniformization

Intuitively, we can obtain a Schottky representation of X by cutting it along g disjoint closed loops such that it stays in one piece and becomes a sphere with $2g$ holes, flatten it onto the complex plane, and build the Schottky group from the Möbius maps that glue the surface back together along its g seams.









Schottky space $\mathfrak{S}_g$

Each generator L_i has a standard form

$$\frac{L_i(w) - a_i}{L_i(w) - b_i} = \lambda_i \frac{w - a_i}{w - b_i},$$

and is completely characterized by its attractive and repulsive fixed points, a_i and b_i, as well as the value of its multiplier λ_i.

By conjugation in $\mathrm{PSL}(2, \mathbb{C})$, one can always put $a_1 = 0$, $b_1 = \infty$, and $a_2 = 1$. A Schottky group for which these conditions hold is called normalized and space of all marked normalized Schottky groups will be called the Schottky space $\mathfrak{S}_g$ of genus g. The Schottky space $\mathfrak{S}_g$ can be viewed as an intermediate moduli space — i.e. $\mathcal{T}_g \to \mathfrak{S}_g \to \mathcal{M}_g$.

Let Σ be a marked normalized Schottky group of genus $g > 1$. The map

$$\Sigma \mapsto (a_3, \ldots, a_g, b_2, \ldots, b_g, \lambda_1, \ldots, \lambda_g) \in \mathbb{C}^{3g-3}$$

establishes a one-to-one correspondence between the Schottky space $\mathfrak{S}_g$ and a connected subset of $\mathbb{C}^{3g-3}$ and can be viewed as defining a coordinate basis in a neighborhood of the origin.

From now on, we will denote the coordinates of $\mathfrak{S}_g$ with $t_1, \ldots, t_{3g-3}$.

Classical Liouville Action for X

Let Σ be a marked normalized Schottky group of rank $g > 1$ which uniformizes the closed Riemann surface X and let $e^{\varphi(w,\bar{w})}|dw|^2$ be the pull-back of the hyperbolic metric on X by the covering map $\pi_\Sigma : \Omega \to X$.

According to [Zograf, Takhtajan '88 b/ Takhtajan, Teo '03], the classical Liouville action for such a compact Riemann surface is defined as

$$S[\varphi] = \iint_{\mathcal{D}} (|\partial_w \varphi|^2 + e^{\varphi})\, \mathrm{d}^2 w + \frac{\sqrt{-1}}{2} \sum_{k=2}^{g} \oint_{C_k} \theta_{L_k^{-1}}(\varphi),$$

where the 1-form $\theta_{L_k^{-1}}(\varphi)$ is given by

$$\theta_{L_k^{-1}}(\varphi) = \left(\varphi - \frac{1}{2}\log|L_k'|^2 - \log|l_k|^2\right)\left(\frac{L_k''}{L_k'}\,\mathrm{d}w - \frac{\overline{L_k''}}{\overline{L_k'}}\,\mathrm{d}\bar{w}\right).$$

This classical Liouville action is independent of the choice of $\mathcal{D}$ and determines a smooth function on $\mathfrak{S}_g$.

Complex Geometry of $\mathfrak{S}_g$

In [Zograf, Takhtajan '88 b], the authors used a mathematically rigorous procedure for varying $S[\varphi]$ with respect to the moduli and were able to prove the following two results:

1. Let $\{t_1, \ldots, t_{3g-3}\}$ denote the coordinates on $\mathfrak{S}_g$ and let $\mathrm{d}t_1, \ldots, \mathrm{d}t_{3g-3}$ be the corresponding cotangent vector fields. If ∂ denotes the (1,0) component of de Rham differential on $\mathfrak{S}_g$, the classical Liouville action satisfies $\partial S[\varphi] = 2R$ where
$$R = -\pi \sum_{i=1}^{3g-3} c_i \, \mathrm{d}t_i \, ,$$
is a (1,0)-form on $\mathfrak{S}_g$ and c_is are the so-called accessory parameters associated with (Fushian) uniformization of $\Omega \cong \mathbb{H}/\mathcal{N}$.
2. If ∂ and $\bar{\partial}$ denote the (1,0) and (0,1) components of de Rham differential on $\mathfrak{S}_g$, we have:
$$\bar{\partial}\partial S = -2\sqrt{-1}\, \omega_{WP}.$$

Moreover, It was proved by Krasnov and Takhtajan-Teo that Liouville action satisfies holographic principle: it is a regularized limit of the hyperbolic volume of a 3D handlebody which has X as its conformal boundary.

Classical Correlation Functions of Branch Point Vertex Operators

Classical Correlation Functions of Branch Point Vertex Operators

Once the classical Liouville action is defined, the quantity $\exp(-S[\varphi])$ will play the role of partition function of classical Liouville thery on X. [Takhtajan '93/ Takhtajan '94 a&b/ Takhtajan '96/ Takhtajan, Teo '06]

However, objects of fundamental importance in classical LFT are given by the correlation functions of vertex operators $V_\alpha(x) = e^{\alpha\varphi(x)}$. These are primary operators of conformal dimension $\Delta = \alpha(2-\alpha)$.

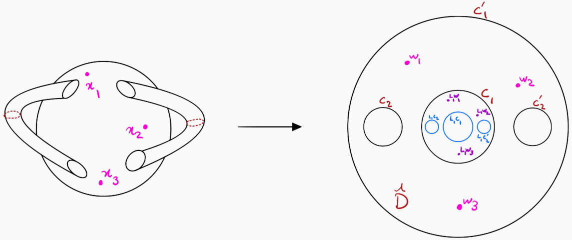
In the classical limit, correlation functions $\langle V_{\alpha_1}(x_1)\cdots V_{\alpha_n}(x_n)\rangle$ are dominated by the extremum of the classical Liouville action with insertion of sources $\sum_{i=1}^{n}\alpha_i\varphi(x_i)$. This then introduces δ-function type singularities on the right hand side of Liouville equation:

$$\partial_u\partial_{\bar{u}}\varphi = \frac{1}{2}e^{\varphi} - \pi\sum\alpha_i\,\delta(u-x_i).$$

Form this point of view, the classical correlation functions $\langle V_{\alpha_1}(x_1)\cdots V_{\alpha_n}(x_n)\rangle$ are given by $\exp(-\mathscr{S}_{\boldsymbol{\alpha}}[\varphi])$ where $\mathscr{S}_{\boldsymbol{\alpha}}[\varphi]$ denotes the classical Liouville action on a Riemann surface with conical singularities x_i of angles $2\pi(1-\alpha_i)$. When $\alpha_i = 1-\frac{1}{m_i}$ $(2\leq m_i\leq\infty)$ the problem of calculating $\langle V_{\alpha_1}(x_1)\cdots V_{\alpha_n}(x_n)\rangle$ reduces to the study of classical Liouville action $\mathscr{S}_{\boldsymbol{m}}[\varphi]$ on a (possibly punctured) Riemann orbisurface O (see also [Park, Takhtajan, Teo '15]).

For our purposes, it is sufficient to view the orbifold Riemann surface O as a underlying Riemann surface X together with n weighted "marked points" $x_1, \ldots, x_n$; the weights $m_1, \ldots, m_n$ will be called the orders of isotropy and the Riemann orbisurface O is said to have the signature $(g, n; m_1, \ldots, m_n)$.

Now, consider the covering map $\pi_\Sigma : \Omega \to X$. By inserting singular points of the same order at the locations corresponding to all pre-images $w_j \in \pi_\Sigma^{-1}(x_i)$ of each marked point x_i $(i = 1, \ldots, n)$, we get a planar orbifold Riemann surface $\overset{\curlywedge}{\Omega}$ which covers O — i.e. $O \cong \overset{\curlywedge}{\Omega}/\Sigma$. We will also denote the restriction of $\overset{\curlywedge}{\Omega}$ to the fundamental domain with $\overset{\curlywedge}{\mathcal{D}}$.

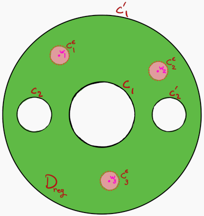


Let us define a generalized Schottky space $\mathfrak{S}_{g,n}(\boldsymbol{m})$ as a holomorphic fibaration $\jmath : \mathfrak{S}_{g,n}(\boldsymbol{m}) \to \mathfrak{S}_g$ with fibers that are configuration spaces of n labeled points (with orders $m_1, \ldots, m_n$). In the neighborhood of the origin, coordinates $t_1, \ldots, t_{3g-3+n}$ of $\mathfrak{S}_{g,n}(\boldsymbol{m})$ are given by $(a_3, \ldots, a_g, b_2, \ldots, b_g, \lambda_1, \ldots, \lambda_g, w_1, \ldots, w_n) \in \mathbb{C}^{3g-3+n}$.

If $\chi(O) = \chi(X) - \sum(1 - \frac{1}{m_i}) < 0$, $\mathbb{H}$ is the universal cover of O and $\overset{\curlywedge}{\Omega}$ itself will admit $\mathbb{H}$ as its universal cover; we denote this covering by $J : \mathbb{H} \to \overset{\curlywedge}{\Omega}$. The covering map J effectively describes the Fuchsian uniformization of $\overset{\curlywedge}{\Omega}$ and its behavior near marked points will play an essential role in our study.

In order to define the appropriate classical Liouville action for O, we have to integrate on $\overset{\curlywedge}{\mathcal{D}}$ instead of on $\mathcal{D}$. Therefore, one needs to regularize the area integral which diverges due to the asymptotic behavior of φ near marked points $w_i \in \overset{\curlywedge}{\mathcal{D}}$.

We do this in the same way as in genus 0 case: [Zograf, Takhtajan '01]



$$S_{\boldsymbol{m}}[\varphi] = S_{\boldsymbol{m}}(\mathcal{D}; w_1, \ldots, w_n) = S_{\mathcal{D}_{\text{reg}}}[\varphi] + \frac{\sqrt{-1}}{2} \sum_{k=2}^{g} \oint_{C_k} \theta_{L_k^{-1}}(\varphi),$$

where

$$S_{\mathcal{D}_{\text{reg}}}[\varphi] =$$

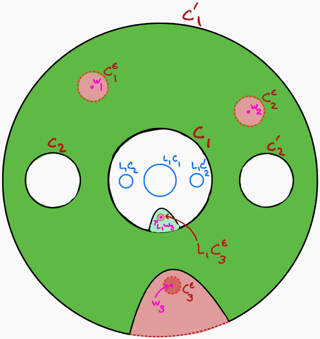
$$\lim_{\epsilon \to 0^+} \Bigg(\iint_{\mathcal{D}_\epsilon} \left(|\partial_w \varphi|^2 + e^{2\varphi}\right) \mathrm{d}^2 w + \frac{\sqrt{-1}}{2} \sum_{j=1}^{n_e} \left(1 - \frac{1}{m_j}\right) \oint_{C_j^\epsilon} \varphi \left(\frac{\mathrm{d}\bar{w}}{\bar{w} - \bar{w}_j} - \frac{\mathrm{d}w}{w - w_j} \right)$$

$$- 2\pi \sum_{j=1}^{n_e} \left(1 - \frac{1}{m_j}\right)^2 \log \epsilon + 2\pi n_p \big(\log \epsilon + 2 \log |\log \epsilon| \big) \Bigg).$$

Anomaly of $S_{\boldsymbol{m}}[\varphi]$

The above regularization procedure, provides a sort of anomaly for the Liouville action which means that $S_{\boldsymbol{m}}[\varphi]$ depends on the choice of representatives in $\Sigma \cdot \{w_1, \ldots, w_n\}$ and no longer determines a function on the Schottky space $\mathfrak{S}_{g,n}(\boldsymbol{m})$. In particular, we have:

$$S_{\boldsymbol{m}}(\tilde{\mathcal{D}}; w_1, \ldots, L_k w_i, \ldots, w_n) - S_{\boldsymbol{m}}(\mathcal{D}; w_1, \ldots, w_n) = \pi \Delta_i \log |L'_k(w_i)|^2.$$



Geometric Meaning of $S_{\boldsymbol{m}}[\varphi]$

The geometric meaning of the above statement is that regularized Liouville action $S_{\boldsymbol{m}}[\varphi]$ determines a Hermitian metric $e^{S_{\boldsymbol{m}}[\varphi]/\pi}$ in the holomorphic $\mathbb{Q}$-line bundle $\mathcal{L} := \mathcal{L}_1^{\Delta_1} \otimes \cdots \otimes \mathcal{L}_n^{\Delta_n}$ over $\mathfrak{S}_{g,n}(\boldsymbol{m})$ where $\mathcal{L}_i$ denotes the i-th relative cotangent line bundle. [B.T., Naseh,Allameh '23]

Then, the following two statement are true: [B.T., Naseh,Allameh '23]

1. In a local holomorphic frame, canonical connection on the Hermitian $\mathbb{Q}$-line bundle $(\mathcal{L}, e^{S_{\boldsymbol{m}}[\varphi]/\pi})$ is given by

$$\frac{1}{\pi}\partial S_{\boldsymbol{m}} = -2 \sum_{i=1}^{3g-3+n} c_i \, \mathrm{d}t_i \, .$$

2. The first Chern form of the Hermitian $\mathbb{Q}$-line bundle $(\mathcal{L}, e^{S_{\boldsymbol{m}}[\varphi]/\pi})$ is given by

$$\mathrm{c}_1(\mathcal{L}, e^{S_{\boldsymbol{m}}[\varphi]/\pi}) = \frac{1}{\pi^2}\omega_{WP}.$$

Kähler Potentials for TZ Metrics

Let $\mathcal{L}_i$ be the i-th tautological line bundle on $\mathfrak{S}_{g,n}(\boldsymbol{m})$ and consider the covering map $J : \mathbb{H} \to \overset{\curlywedge}{\Omega}$. Since $J \circ \beta_k = L_k \circ J$, the marked points $w_1, \ldots, L_k w_i, \ldots, w_n$ correspond to the fixed points $z_1, \ldots, \beta_k z_i, \ldots, z_n$, and the first coefficient in the expansion of $J(z)$ at the equivalent fixed point $\beta_k z_i$ is $L_k'(w_i) J_1^{(i)}$. Correspondingly, $h_i = |J_1^{(i)}|^2$ gets replaced by $h_i |L_k'(w_i)|^2$. Geometrically, this means that the quantities h_i determine Hermitian metrics in the holomorphic line bundles $\mathcal{L}_i$ for all $i = 1, \ldots, n$.

Then, the following two statement are true: [Park, Takhtajan, Teo '15/ Takhtajan, Zograf '18/ B.T., Naseh, Allameh '23]

1. In a local holomorphic frame canonical connection on the Hermitian line bundle $(\mathcal{L}_i, h_i)$ is given by

$$\partial \log h_i = \frac{-2}{\pi} \sum_{j=1}^{3g-3+n} d_{i,j} \, \mathrm{d}t_j \, .$$

2. The first Chern form of the Hermitian line bundle $(\mathcal{L}_i, h_i)$ is given by

$$\mathrm{c}_1(\mathcal{L}_i, h_i) = \frac{m_i}{2\pi} \omega_{TZ,i}^{ell} \quad (m_i < \infty) \text{ and } \mathrm{c}_1(\mathcal{L}_i, h_i) = \frac{4}{3} \omega_{TZ,i}^{cusp} \quad (m_i = \infty).$$

Kähler Potentials for TZ Metrics

Now, let us define $H := \Pi_{i=1}^{n} h_i^{\Delta_i}$. Clearly, H defines a Hermitian metric in the holomorphic $\mathbb{Q}$-line bundle $\mathcal{L} := \mathcal{L}_1^{\Delta_1} \otimes \cdots \otimes \mathcal{L}_n^{\Delta_n}$ over $\mathfrak{S}_{g,n}(\boldsymbol{m})$.

Then, the previous statements about connections and Chern forms on line bundles $\mathcal{L}_i$ can be written as: [Park, Takhtajan, Teo '15/ Takhtajan, Zograf '18/ B.T., Naseh, Allameh '23]

1. In a local holomorphic frame, the canonical connection on the Hermitian $\mathbb{Q}$-line bundle $(\mathcal{L}, H)$ is given by

$$\partial \log H = \frac{-2}{\pi} \sum_{j=1}^{3g-3+n} \underbrace{\sum_{i=1}^{n} \Delta_i d_{i,j}}_{d_j} \mathrm{d}t_j \,.$$

2. The first Chern form of the $\mathbb{Q}$-Hermitian line bundle $(\mathcal{L}, H)$ is given by

$$\mathrm{c}_1(\mathcal{L}, H) = \frac{4}{3}\omega_{TZ}^{cusp} + \frac{1}{2\pi} \sum_{i=1}^{n_e} \Delta_i m_i \omega_{TZ,i}^{ell}.$$

Main Results

Classical Liouville Action on $\mathfrak{S}_{g,n}(\boldsymbol{m})$

Combining the previous discussions, we conclude that the combination $\mathscr{S}_{\boldsymbol{m}}[\phi] := S_{\boldsymbol{m}}[\phi] - \pi \log H$ determines a smooth real-valued function on $\mathfrak{S}_{g,n}(\boldsymbol{m})$! This means that $\exp(-\mathscr{S}_{\boldsymbol{m}}[\phi])$ gives the correct classical contribution to the correlation function of heavy Liouville vertex operators.

Theorem (B.T., Naseh, Allameh)
Let ∂ and $\bar{\partial}$ be the (1,0) and (0,1) components of the de Rham differential on $\mathfrak{S}_{g,n}(\boldsymbol{m})$. The following statements hold:

1. *The function $\mathscr{S}_{\boldsymbol{m}}[\phi]$ on $\mathfrak{S}_{g,n}(\boldsymbol{m})$ satisfies $\partial\mathscr{S}_{\boldsymbol{m}}[\phi] = 2\mathscr{R}$ where*

$$\mathscr{R} = \sum_{i=1}^{3g-3+n} (-\pi c_i + d_i)\, \mathrm{d}t_i \,,$$

   is a (1,0)-form on $\mathfrak{S}_{g,n}(\boldsymbol{m})$.

2. *The function $-\mathscr{S}_{\boldsymbol{m}}[\phi]$ on $\mathfrak{S}_{g,n}(\boldsymbol{m})$ is a potential for the special combination of Weil-Petersson and Takhtajan-Zograf metrics:*

$$-\bar{\partial}\partial\mathscr{S}_{\boldsymbol{m}}[\phi] = 2\sqrt{-1}\left(\omega_{WP} - \frac{4\pi^2}{3}\omega_{TZ}^{cusp} - \frac{\pi}{2}\sum_{i=1}^{n_e} \Delta_i m_i \omega_{TZ,i}^{ell}\right).$$

Questions?

Thank you!