Non-Relativistic Conformal Field Theory

M.M Ahmadi Jahmani

Physics Department of Ferdowsi University of Mashhad

December 14, 2022

Non-Relativistic Conformal Field Theory

- Conformal field theories (CFTs) form a special class of relativistic quantum field theories, where the poincaré symmetry group is enlarged to the group of conformal transformations.[1]

- CFTs appear as the description of the fixed point of the renormalization group flow in a relativistic quantum field theory.[2]

Non-Relativistic Conformal Field Theory

- Non-relativistic quantum systems with two-body bound state energies tuned close to threshold seem to describe a variety of physical problems.[3]

- This tuning may be accidental, as it is in the effective theories that describe the few-body strong interactions of nucleons at low momentum transfers, or arranged by experimental manipulation, as with cold atoms placed in an external confining trap.[3]

Non-Relativistic Conformal Field Theory

- As the binding energy approaches threshold, the scattering length diverges and the precise nature of the two-body interactions becomes irrelevant.[3]

- In this limit, the dynamics can be formulated in terms of local non-relativistic scale invariant field theories (NRCFTs), in which the usual Galilean invariance of the interactions is enhanced to a non relativistic conformal symmetry known as Schrödinger symmetry.[3]

Non-Relativistic Conformal Field Theory

- CFTs appear as the description of the fixed point of the renormalization group flow; in non-relativistic regime ,nonrelativistic CFT describing fermions at unitarity in the fixed point.[1]

- In the unitary limit (or fermions at unitary), the thermodynamic properties have both bosonic and fermionic features.[4]

- Thus in the unitary limit we have $c_0 \to \infty$.

Non- Relativistic Quantum Field theory

- the Lagrange density of the Schrödinger field[5]:

$$\wp = i\hat{\psi}^{\dagger}_{\alpha}(x,t)\partial_t \hat{\psi}_{\alpha}(x,t) - \frac{1}{2}\nabla \hat{\psi}^{\dagger}_{\alpha}(x,t).\nabla \hat{\psi}_{\alpha}(x,t)$$

$$= i\hat{\psi}^{\dagger}_{\alpha}(x,t)\partial_t \hat{\psi}_{\alpha}(x,t) - \frac{1}{2}\left|\nabla \hat{\psi}_{\alpha}(x,t)\right|^2$$

$$\hbar = m = 1 \Rightarrow non-relativistic\ natural\ units$$

Non- Relativistic Quantum Field theory

- **Field content:**

- Non-relativistic spin-zero particle[5,6]:

$$\hat{\phi}\left(\vec{x},t\right)=\int\frac{d\vec{k}}{\left(2\pi\right)^{d}}\ \hat{a}\left(\vec{k}\right)e^{-i\left(\omega_k t-\vec{k}.\vec{x}\right)}$$

$$\hat{\phi}^{*}\left(\vec{x},t\right)=\int\frac{d\vec{k}}{\left(2\pi\right)^{d}}\ \hat{a}^{\dagger}\left(\vec{k}\right)e^{+i\left(\omega_k t-\vec{k}.\vec{x}\right)}$$

Non- Relativistic Quantum Field theory

- **Field content:[6]**
- Spin- $^1/_2$ particle (fermions)

$$\hat{\psi}_{\uparrow}(x,t)=\int\frac{d\vec{k}}{(2\pi)^d}\,\hat{b}\left(\vec{k}\right)e^{-i\left(\omega_k t-\vec{k}.\vec{x}\right)}$$

$$\hat{\psi}_{\uparrow}^{\dagger}(x,t)=\int\frac{d\vec{k}}{(2\pi)^d}\,\hat{b}^{\dagger}\left(\vec{k}\right)e^{+i\left(\omega_k t-\vec{k}.\vec{x}\right)}$$

$$\hat{\psi}_{\downarrow}(x,t)=\int\frac{d\vec{k}}{(2\pi)^d}\,\hat{d}\left(\vec{k}\right)e^{-i\left(\omega_k t-\vec{k}.\vec{x}\right)}$$

$$\hat{\psi}_{\downarrow}^{\dagger}(x,t)=\int\frac{d\vec{k}}{(2\pi)^d}\,\hat{d}^{\dagger}\left(\vec{k}\right)e^{+i\left(\omega_k t-\vec{k}.\vec{x}\right)}$$

Non- Relativistic Quantum Field theory

- We start by considering two-particle scattering of fermions with a four-fermion point interaction. The Lagrangian of the system (the ideal fermion gas) is given by[6,7]:

$$\wp = \sum_{\alpha=\uparrow,\downarrow} \hat{\psi}_{\alpha}^{\dagger} \left(i\partial_t + \frac{1}{2}\partial_x^2 \right) \hat{\psi}_{\alpha} - c_0 \hat{\psi}_{\uparrow}^{\dagger} \hat{\psi}_{\downarrow}^{\dagger} \hat{\psi}_{\downarrow} \hat{\psi}_{\uparrow}$$

Non- Relativistic Quantum Field theory

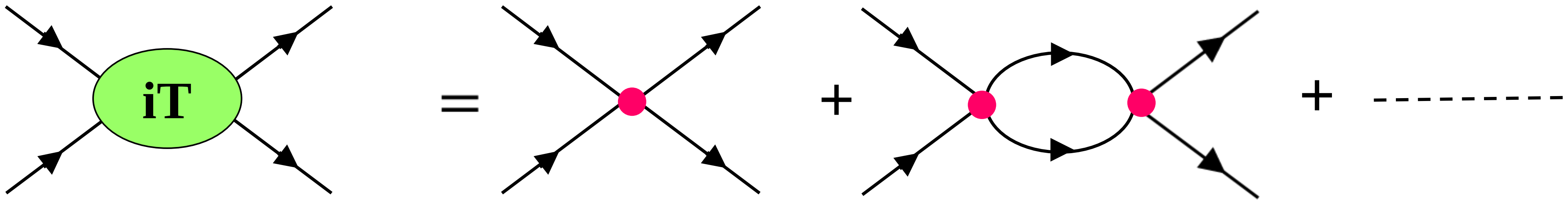
- The fermions have a propagator[7]:

$$iG(\omega, \vec{p}) = \frac{i}{\omega - \varepsilon_p + i\delta}$$

- Where $\varepsilon_p = \frac{p^2}{2m} = \frac{p^2}{2}$.

Non- Relativistic Quantum Field theory

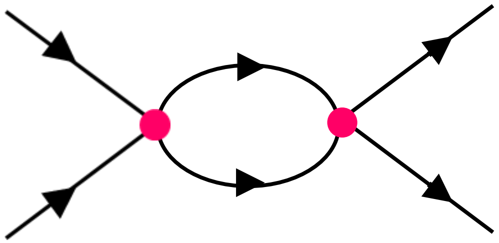
- The T-matrix amplitude for two-particle scattering in vacuum is given by an infinite sum of bubble diagrams, which can be summed as a geometric series[7]:



$$iT(\omega, p) = ic_0 \sum_{n=0} (iM)^n = \frac{ic_0}{1 - iM}$$

Non- Relativistic Quantum Field theory

- Where $ic_0 \times iM$ is one-bubble diagram, i.e. the second diagram that is[7]:



- Hence

$$iM = c_0 \int \frac{dk}{(2\pi)^d} \frac{1}{2\varepsilon_k - \omega + \frac{\varepsilon_p}{2} - i\delta}$$

Non- Relativistic Quantum Field theory

- Hence

$$T\left(\omega, p\right)^{-1} = \frac{1}{c_0} - \int \frac{d\vec{k}}{\left(2\pi\right)^d} \frac{1}{2\varepsilon_k - \omega + \frac{\varepsilon_p}{2} - i\delta}$$

- The integral in the relation above is divergent and needs to be regularized. [7]

Non- Relativistic Quantum Field theory

- Thus in the unitary limit we have $c_0 \to \infty$; therefore in this limit we have $T\left(0,0\right)^{-1} = 0$.[7]

- The integration over $\vec{k}$ can be evaluated explicitly:

$$iT\left(\omega, p\right) = -\Gamma\left(1-\frac{d}{2}\right)\left(\frac{1}{4\pi}\right)^{\frac{d}{2}}\left(-\omega+\frac{\varepsilon_p}{2}-i\delta\right)^{\frac{d}{2}-1}$$

Non- Relativistic Quantum Field theory

- Substituting $d = 4 - \varepsilon$ and expanding in terms of ε , the T-matrix near four spatial dimensions becomes:

$$iT\left(\omega, \vec{p}\right) = ig^2 \frac{1}{\omega - \frac{\varepsilon_p}{2} + i\delta}$$

ε=4-d <<1 : d=spatial dimensions

- If we define $g^2 = 8\pi^2\varepsilon$; So that the g is effective coupling between fermions.[7]

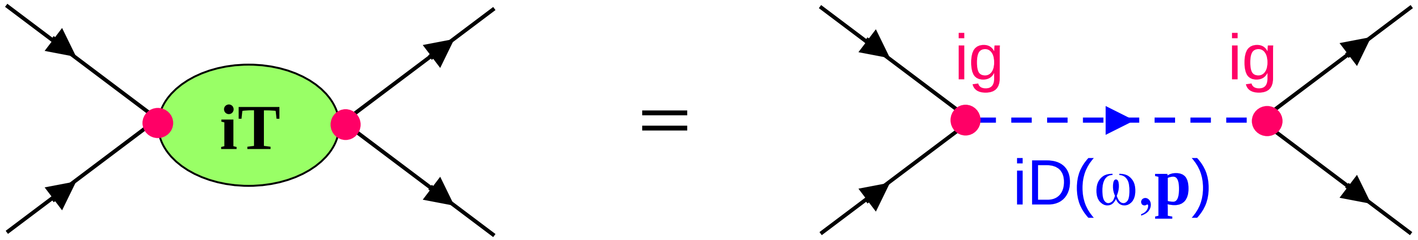
Non- Relativistic Quantum Field theory

- And

$$D(\omega,\vec{p})=\left(\omega-\frac{\varepsilon_p}{2}+i\delta\right)^{-1}$$

- The function $D(\omega,\vec{p})$ is the propagator of a particle with mass 2. It is natural to interpret this particle a bound state of two fermions at threshold. We will refer to this particle simply as the boson.[7]

Non- Relativistic Quantum Field theory



$g^2 = (8\pi^2 \boldsymbol{\varepsilon})$

Non- Relativistic Quantum Field theory

- Fermion statistics prohibit condensation, but fermions can get around it by forming fermion pair (known as Cooper pairs).[6]

- Since fermion pairs are boson-like particles, while noninteracting elementary bosons are known to condense all into the same state.[6]

- Furthermore, we want to describe things in terms of fermion pairs, and actually we'll kill two birds with one stone through a Hubbard-Stratonovich transformation. [6,7]

Non- Relativistic Quantum Field theory

- By making use of the identity:

$$|a-b|^2 = |a|^2 + |b|^2 - a^* b - b^* a$$

- and taking $a = \phi$ and $b = g\psi_\downarrow \psi_\uparrow$, we rewrite the interaction term as:

$$g\left|\hat{\psi}_\downarrow \hat{\psi}_\uparrow\right|^2 = \frac{1}{g}\left|\hat{\phi} - g\hat{\psi}_\downarrow \hat{\psi}_\uparrow\right|^2 - \frac{1}{g}\left|\hat{\phi}\right|^2 + \hat{\phi}^* \hat{\psi}_\downarrow \hat{\psi}_\uparrow + \hat{\phi}\hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow^\dagger$$

Non- Relativistic Quantum Field theory

- Choosing $\hat{\phi} = g\hat{\psi}_{\downarrow}\hat{\psi}_{\uparrow}$ the first term on the right hand side cancels, so that we only have quadratic terms on ψ .[7]

- More than just doing a mathematical trick, we are essentially defining a field describing pairs of electrons with opposite spins! [7]

Non- Relativistic Quantum Field theory

- The Lagrangian density describing fermions at unitarity is[1,8]:

$$\wp = i\hat{\psi}_{\sigma}^{\dagger}\,\partial_t\hat{\psi}_{\sigma} - \frac{1}{2}\left|\nabla\,\hat{\psi}_{\sigma}\right|^2 + i\hat{\phi}^{*}\,\partial_t\hat{\phi} - \frac{1}{4}\left|\nabla\,\hat{\phi}\right|^2 + g\hat{\psi}_{\uparrow}^{\dagger}\hat{\psi}_{\downarrow}^{\dagger}\hat{\phi} + g\hat{\psi}_{\downarrow}\hat{\psi}_{\uparrow}\hat{\phi}^{*}$$

$$\hat{\phi}(x) = g\,\hat{\psi}_{\downarrow}(x)\hat{\psi}_{\uparrow}(y)$$

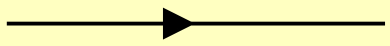
$$m_{\phi} = m_{\psi_{\uparrow}} + m_{\psi\downarrow} \qquad ;$$

$$\dim[\hat{\psi}] = \dim[\hat{\psi}^{\dagger}] = \frac{d}{2}; \qquad \dim[\hat{\phi}] = \dim[\hat{\phi}^{*}] = 2$$

Non- Relativistic Quantum Field theory

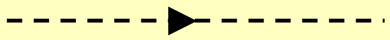
- Feynman rules[6,7]

fermion propagators:



$$iG\left(\omega, \vec{p}\right) = \frac{i}{\omega - \varepsilon_p + i\delta}$$

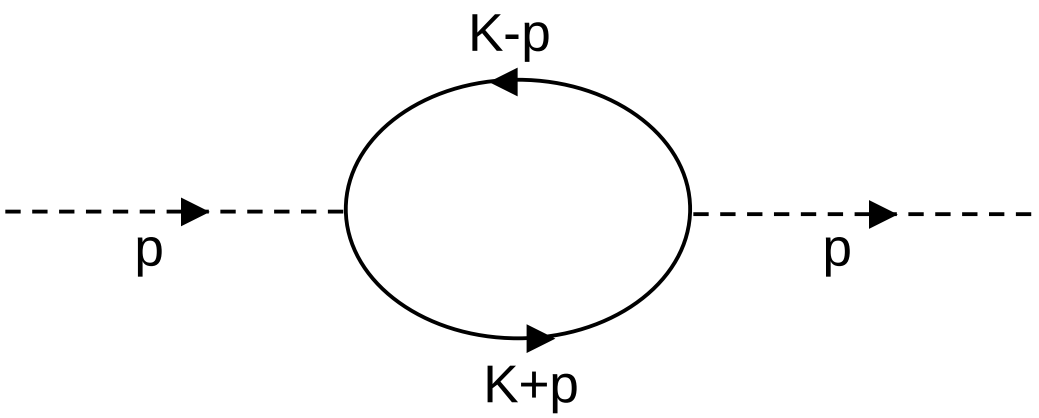
boson propagators:



$$D\left(\omega, \vec{p}\right) = \frac{1}{\omega - \frac{\varepsilon_p}{2} + i\delta}$$

Non- Relativistic Quantum Field theory

- The boson self-energy



Non- Relativistic Quantum Field theory

- The boson self-energy[1]:

$$\Sigma\left(p\right)=ig^{2}\int_{e^{-s}\Lambda}^{\Lambda}\frac{d\vec{K}}{\left(2\pi\right)^{d}}\int\frac{d\omega}{2\pi}G\left(k+\frac{p}{2}\right)G\left(k-\frac{p}{2}\right)$$

$$=-\frac{g^{2}}{8\pi^{2}}\left(p^{0}-\frac{p^{2}}{4}\right)\ln\frac{\Lambda}{e^{-s}\Lambda}$$

Non- Relativistic Quantum Field theory

- The wave-function renormalization of ϕ :

$$Z_\phi = 1 - \frac{g^2}{8\pi^2} s$$

- The anomalous dimension of ϕ is:

$$\gamma_\phi = -\frac{1}{2}\frac{\partial \ln Z_\phi}{\partial s} = \frac{g^2}{16\pi^2}$$

Non- Relativistic Quantum Field theory

- The β function that governs the running of g

$$\beta(g) = \frac{\partial g}{\partial s}$$

- Then we have:

$$\beta(g) = \left(2 - \frac{d}{2} - \gamma_{\phi}\right) g = \frac{\varepsilon g}{2} - \frac{g^3}{16\pi^2}$$

$$d = 4 - \varepsilon$$

Non- Relativistic Quantum Field theory

- There is a fixed point located at:

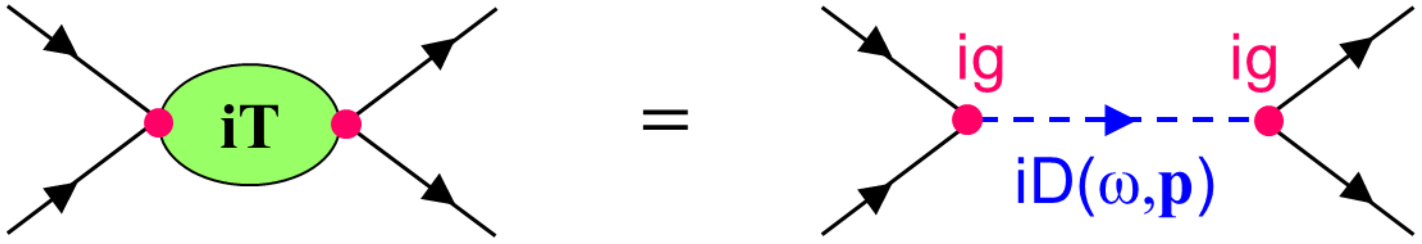
$$\beta(g) = 0 \Rightarrow g^2 = 8\pi^2\varepsilon$$

- At this fixed point, the theory is a nonrelativistic CFT describing fermions at unitarity.

- In the unitary limit, the thermodynamic properties have both bosonic and fermionic features.

Non- Relativistic Quantum Field theory

- At this fixed point, the phase transition occurs. This means that the two fermionic d.o.f turns into one bosonic d.o.f. [6]



$$g^2 = (8\pi^2 \varepsilon)$$

Conclusion

- we have seen that it is possible to perform a perturbative expansion of a Fermions in the vicinity of unitarity using an ε-expansion; that in four spatial dimensions, the ground state of the unitary Fermi gas is a gas of non-interacting bosons.

Thank you

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