Critical O(N) Vector Model

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Contents

What is the effective action?

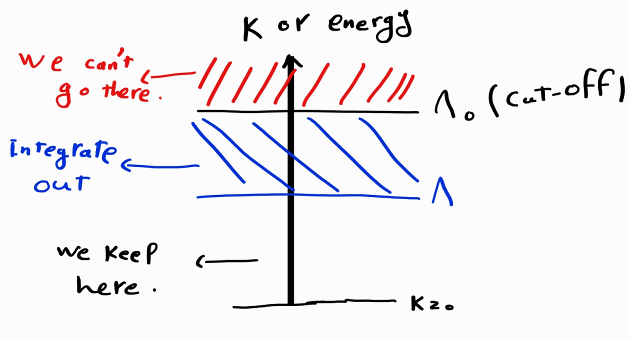
consider general form of scalar field theory:

$$S_{\Lambda_0} = \int d^d x \left[\frac{1}{2}(\partial^\mu \phi \partial_\mu \phi) + \sum_i \Lambda_0^{d-d_i} g_{0i} O_i(x) \right]$$

where Λ_0 is cutoff, $d_i = [O_i]$, d is the space-time dimension and $O_i(x)$ is any operator made of field and it's derivatives. for the partition function we have:

$$\mathcal{Z} = \int_{c^\infty(M) \leq \Lambda_0} \mathcal{D}\phi e^{-S_{\Lambda_0}(\phi)/\hbar}$$

Now we want to break modes into two parts: high-energy modes and low-energy modes:



$$\phi(x) = \int_{|p|\leq\Lambda} \frac{d^dp}{(2\pi)^d} e^{ip.x}\tilde{\phi}(p) + \int_{\Lambda<|P|\leq\Lambda_0} \frac{d^dp}{(2\pi)^d} e^{ip.x}\tilde{\phi}(p)$$
$$:= \Phi + \chi$$

where Φ is Low-energy mode and χ is High-energy mode. also we know $\Lambda = b\Lambda_0, \quad 0 < b < 1$

Now by integrating out high-energy modes, we can find effective action:

$$\mathcal{Z} = \int \mathcal{D}\Phi \int \mathcal{D}\chi e^{\frac{-S_{\Lambda_0}}{\hbar}} \quad \rightarrow \quad S_{\Lambda}^{eff}(\Phi) := -\hbar \ln \left[\int \mathcal{D}\chi \exp(\frac{-S_{\Lambda_0}}{\hbar}) \right]$$

also we can write:

$$S_{\Lambda}^{eff}(\Phi) = S^0(\Phi) + S_{\Lambda}^{int}(\Phi)$$

where

$$S^0(\Phi) = \int d^d x (\frac{1}{2}(\partial\Phi)^2 + \frac{1}{2} m^2 \Phi^2)$$

$$S_{\Lambda}^{int}(\Phi) = -\ln \int \mathcal{D}\chi exp(-S^0(\chi) - S_{\Lambda_0}^{int}(\Phi + \chi))$$

note that:

$$S_{\Lambda_0}(\Phi + \chi) = S^0(\Phi) + S^0(\chi) + S_{\Lambda_0}^{int}(\Phi + \chi)$$

Beta function and Anomalous dimension

now we can work with this partition function and we can generalize our S_Λ^{eff}:

$$\mathcal{Z}_\Lambda = \int \mathcal{D}\Phi e^{\frac{-S_\Lambda^{eff}(\Phi)}{\hbar}}$$

$$S_\Lambda^{eff}(\Phi) = \int d^d x\left(\frac{Z_\Lambda}{2}(\partial\Phi)^2 + \sum_i \Lambda^{d-d_i} Z_\Lambda^{\frac{n_i}{2}} g_i(\Lambda) O_i\right)$$

where Z_Λ is Renormalization factor: $\phi^{nor} := Z_\Lambda^{\frac{1}{2}}\Phi$. we define Beta function as:

$$\beta_i(g) := \Lambda \frac{\partial g_i}{\partial \Lambda}$$

more over we define anomalous dimension as (from now we rewrite $\Phi \to \phi$ for convension):

$$\gamma_\phi := \frac{-1}{2}\Lambda \frac{\partial \ln Z_\Lambda}{\partial \Lambda}$$

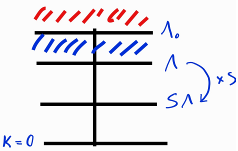
Callan–Symanzik equation

For n-point Function we have:

$$\langle\phi(x_1)\ldots\phi(x_n)\rangle = Z_{\Lambda}^{\frac{-n}{2}}\langle\phi^{nor}(x_1)\ldots\phi^{nor}(x_n)\rangle \equiv Z_{\Lambda}^{\frac{-n}{2}}\Gamma_{\Lambda}^{(n)}(x_1,\ldots,x_n;g_i(\Lambda))$$

actually we can go deep in to lower energies using parameter $s < 1$ (if all modes have energies $<< \Lambda$):

$$Z_{s\Lambda}^{\frac{-n}{2}}\Gamma_{s\Lambda}^{(n)}(x_1,\ldots,x_n;g_i(s\Lambda)) = Z_{\Lambda}^{\frac{-n}{2}}\Gamma_{\Lambda}^{(n)}(x_1,\ldots,x_n;g_i(\Lambda))$$



Callan–Symanzik equation is:

$$\Lambda\frac{d\Gamma_{\Lambda}^{(n)}}{d\Lambda} = (\Lambda\frac{\partial}{\partial\Lambda} + \beta_i\frac{\partial}{\partial g_i} + n\gamma_{\phi})\Gamma_{\Lambda}^{(n)} = 0$$

this eq. shows that all functions change in a manner which Γ remains unchanged.

Rescale and Scaling dimension

We change the scale of x as $x^\mu \to x'^\mu := sx^\mu$. we know for $r << \xi$ (ξ is correlation-length) 2-point function scales as:

$$\langle \phi(x)\phi(y)\rangle \sim \frac{1}{|x-y|^{d-2}}$$

so under s-rescaling we have $\phi(sx) \sim s^{-\frac{(d-2)}{2}}\phi(x)$. because of the fact that length dim. is inverse of energy dim., we have $\Lambda \to \frac{\Lambda}{s}$ so we achieve $S\Lambda \to s\frac{\Lambda}{s} = \Lambda$.

$$\Gamma^{(n)}_\Lambda(x_i) = \left(\frac{Z_\Lambda^{\frac{n}{2}}}{Z_{s\Lambda}^{\frac{n}{2}}}\right)\Gamma^{(n)}_{s\Lambda}(x_i) = \left(s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{\frac{n}{2}}\Gamma^{(n)}_\Lambda(sx_i)$$

Also, we can have:

$$\Gamma^{(n)}_\Lambda\left(\frac{x_i}{s}\right) = \left(s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{\frac{n}{2}}\Gamma^{(n)}_\Lambda(x_i)$$

If we expand $\left(s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{\frac{n}{2}}$ for $s = 1-\delta s$ we can define Scaling-dimension as:

$$\Delta_\phi \equiv \left(\frac{d-2}{2}\right) + \gamma_\phi$$

RG flow

Question: What happen when we change Λ?
before answering this question, we note, if beta function vanishes we have "critical-point". in this point we have $g_i = g_i^*$. we can always have "trivial critical-point" or "Gaussian critical-point" which is $g_i^* = 0$ and our theory is free.
One can show for a theory at critical-point we have:

$$\langle T^{\mu}_{\mu} \rangle = 0$$

This condition is the general condition for "Conformal field theories". so we can say at critical point our QFT is a CFT! (for 2-dim. we can prove it and for d>2 we believe it holds.)

What happen for theories near critical-points?
at $g_i = g_i^* + \delta g_i$ we have $\beta_i = \beta_{ij}\delta g_j + O(\delta g^2)$, so:

$$\Lambda \frac{\partial g_i}{\partial \Lambda}|_{g_i^*+\delta g_i} = \beta_{ij}\delta g_j + O(\delta g^2)$$

we call β_{ij} beta matrix. this matrix has eigenvector σ_i and let its eigenvalue be $\Delta_i - d$. so one finds:

$$\Lambda \frac{\partial \sigma_i}{\partial \Lambda} = \beta_{ij}\sigma_j = (\Delta_i - d)\sigma_j + O(\sigma^2)$$

this leads us to:

$$\sigma_i(\Lambda) = (\frac{\Lambda}{\Lambda_0})^{\Delta_i - d}\sigma_i(\Lambda_0)$$

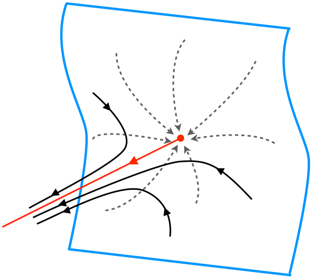
This is "RG flow eq." of operator σ_i.

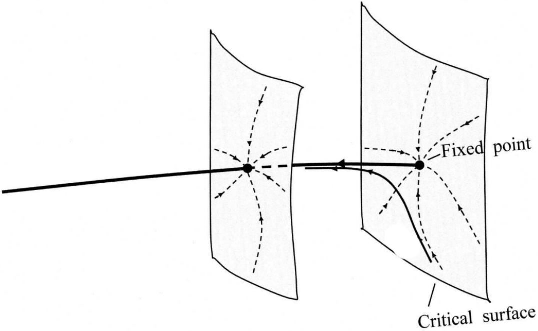
RG Trajectory

There are three possible condition:

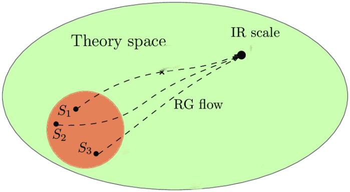
- if $\Delta_i > d$ we have "Irrelevant" operator. by going to IR this fades away and we "flow back" to critical-point.
- if $\Delta_i > d$ we have "Relevant" operator. This operator will survive in IR.
- if $\Delta_i = d$ we have "Marginal" operator. This operator won't change in IR.

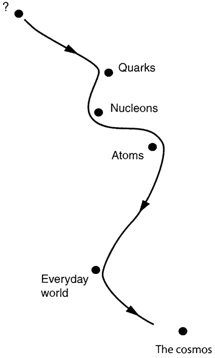
$$\sigma_i(\Lambda) = (\frac{\Lambda}{\Lambda_0})^{\Delta_i - d}\sigma_i(\Lambda_0)$$




Fixed point
Critical surface

Universality class





Polchinski's ERG equation

If we start from:

$$S_{\Lambda}^{int}(\Phi) = -\ln \int \mathcal{D}\chi exp(-S^0(\chi) - S_{\Lambda_0}^{int}(\Phi+\chi))$$

and expand $\exp(-S_{\Lambda_0}^{int}(\phi+\chi))$, by knowing that every χ propagators will give a factor of $\delta\Lambda$ and taking Λ to be small, one can keep zero and second term of above expansion. after few steps we can have Polchinski's equation:

$$-\Lambda \frac{\partial S_{\Lambda}^{int}(\phi)}{\partial \Lambda} = \int d^d x d^d y (\frac{\delta S_{\Lambda}^{int}(\phi)}{\delta \phi(x)} D_{\Lambda}(x,y) \frac{\delta S_{\Lambda}^{int}(\phi)}{\delta \phi(y)} - D_{\Lambda}(x,y) \frac{\delta^2 S_{\Lambda}^{int}(\phi)}{\delta \phi(x) \delta \phi(y)})$$

where D_{Λ} is the propagator of χ. we can rewrite this equation as:

$$\frac{\partial}{\partial t} e^{-S^{int}(\phi)} = -\int d^d x d^d y (D_{\Lambda}(x,y) \frac{\delta^2}{\delta \phi(x) \delta \phi(y)} e^{-S^{int}(\phi)})$$

where $t \equiv \ln(\Lambda)$, we call this parameter "RG time". in AdS/CFT framework, this time is the actual time in AdS bulk.

The local potential approximation

- Although Polchinski's eq. have every possible "exact information" about operators under RG flow, but it super hard to solve it! so people have to deal with some approximations.
- if in $d > 2$ we drop terms with derivative of ϕ we have:

$$S_{\Lambda}^{eff}(\phi) = \int d^d x (\frac{1}{2}(\partial\phi)^2 + V(\phi))$$

$$V(\phi) = \sum_{K} \Lambda^{d-k(d-2)} \frac{g_{2k}}{(2k)!} \phi^{2k}$$

we call this approx. "Local potential approx."

- L.P.A. leads to following eq.:

$$\Lambda \frac{dg_{2K}}{d\Lambda} = (k(d-2) - d) 2 g_{2k} - a \Lambda^{k(d-2)} \frac{\partial^{2k}}{\partial \phi^2 k} \ln(\Lambda^2 + V''(\phi))|_{\phi=0}$$

where we have $a = \frac{1}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2})}$.

Using the last eq. we can have :

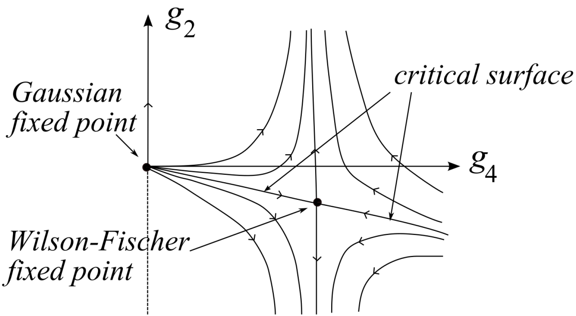
$$k = 1 \to \Lambda \frac{dg_2}{d\Lambda} = -2g_2 - \frac{ag_4}{1+g_2} \qquad \text{Mass term}$$

$$k = 2 \to \Lambda \frac{dg_4}{d\Lambda} = (d-4)g_4 - \frac{ag_6}{1+g_2} + \frac{3ag_4^2}{(1+g_2)^2} \qquad \phi^4 \text{ term}$$

Wilson-Fisher critical point

In $d < 4$ coupling of ϕ^4 theory is relevant so as we go deep inside IR area, we get strongly coupled theory. the trick is we go little bit (ϵ) inside ie. $d = 4 - \epsilon$. using the results of last slide, Wilson and Fisher found two "non-trivial" critical points as:

$$g_2^{WF} = -\frac{1}{6}\epsilon + O(\epsilon^2), \qquad g_4^{WF} = \frac{1}{3a}\epsilon + O(\epsilon^2)$$



The O(N) model

The water vapor phase diagram is described using d=3 Ising model. this model near critical-point is modeled by Euclidean QFT of real scalar field with the action:

$$S = \int d^3x(\frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{4!}\phi^4)$$

As we talked about it, this theory at $d = 3$, as we go deep into IR regime, has strongly coupled interaction term. so we can not use ordinary perturbation techniques!

Do not worry about this issue, we have some solution to study this theory in $d < 4$:

- Wilson-Fisher ϵ expansion.
- $\frac{1}{N}$expansion and large N limit.
- Bootstrap (Mr. Ameri's seminar)

ϵ-expansion

If we consider general case of having N scalar field (real) and show them with vector as:

$$\phi^i = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} \quad i = 1, \ldots, N$$

this field have $O(N)$ invariant action:

$$S = \int d^3x (\frac{1}{2}(\partial\phi^i)^2 + \frac{\lambda}{4}(\phi^i\phi^i)^2)$$

to investigate $d < 4$, as before we go little bit (ϵ) inside ie. $d = 4 - \epsilon$. we calculate β_λ as:

$$\beta_\lambda = -\epsilon\lambda + (N+8)\frac{\lambda^2}{8\pi^2}$$

and the IR fixed point is:

$$\lambda_* = \frac{8\pi^2}{N+8}\epsilon$$

we can go further and calculate anomalous dim.:

$$\gamma_\phi = \frac{N+2}{4(N+8)^2}\epsilon^2 + O(\epsilon^3)$$
$$\gamma_{\phi^2} = \frac{N+2}{N+8}\epsilon + O(\epsilon^2)$$

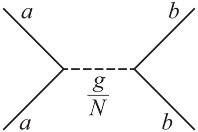
these lead us to scaling dim. for ϕ^i and $\phi^i\phi^i$:

$$\Delta_\phi = \frac{d-2}{2} + \gamma_\phi = 1 - \frac{\epsilon}{2} + \frac{N+2}{4(N+8)^2}\epsilon^2 + O(\epsilon^3)$$
$$\Delta_{\phi^2} = d - 2 + \gamma_{\phi^2} = 2 - \frac{6}{N+8}\epsilon + O(\epsilon^3)$$

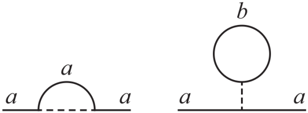
actually now for the case of $d = 3$ we can set $\epsilon = 1$ and experimental and numerical results approve this trick works.

$\frac{1}{N}$ expansion (diagrammatically)

We first change the coupling to $\lambda \to \frac{g}{N}$ and at the end take the limit $N \to \infty$.
for interaction vertex we have:

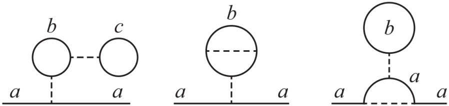


and also for one-loop self energy Feynman diagrams are:



- lfh diagram has a factor $\frac{g}{N}$ so by large N limit this diagram fades away.
- rhs diagram has also $\frac{g}{N}$ factor, but this one has an extra factor N because of sum over indices for loop. so this diagram has total factor g at the end.

also for two loop we have:



only lfh diagram will survive after large N limit. so at every order only diagrams with the number of loops equal the number of order will not die.

you may wonder we did not do any thing special because still we have coupling without $\frac{1}{N}$ factor. we should note that we impose "T'hooft coupling" condition too:

$$N \to \infty, g \to \text{fixed value}(\sim 0), g \equiv \lambda N$$

by this condition we have more smooth diagrams.

$\frac{1}{N}$ expansion (path integral)

the usual technique is to introducing a "Hubbard-Stratonovich" auxiliary field σ as:

$$S = \int d^d x (\frac{1}{2}(\partial\phi^i)^2 + \frac{1}{2}\sigma\phi^i\phi^i - \frac{\sigma^2}{4\lambda})$$

Note that we can alway back to standard action using e.o.m. $\sigma = \lambda\phi^i\phi^i$. we can also drop the last term when we go to IR (in comparison to kinetic term of σ) and work with:

$$S = \int d^d x (\frac{1}{2}(\partial\phi^i)^2 + \frac{1}{2\sqrt{N}}\sigma\phi^i\phi^i)$$

by integrating out fundamental fields ϕ^i we get an effective action:

$$\mathcal{Z} = \int \mathcal{D}\phi\mathcal{D}\sigma e^{-\int d^d x (\frac{1}{2}(\partial\phi^i)^2 + \frac{1}{2\sqrt{N}}\sigma\phi^i\phi^i)}$$

$$= \int \mathcal{D}\sigma e^{\frac{1}{8N}\int d^d x d^d y (\sigma(x)\sigma(y)\langle\phi^i(x)\phi^i(x)\phi^j(y)\phi^j(y)\rangle_0 + O(\sigma^3))}$$

using last path integral one can find Propagator for ϕ^i fields and for σ. Using them and after computing one-loop correction to ϕ^i field, one get:

$$\Delta_\phi = \frac{d-2}{2} + \frac{1}{N}\eta_1 + \frac{1}{N^2}\eta_2$$

$$\Delta_\sigma = 2 + \frac{1}{N}\frac{4(d-1)(d-2)}{(d-4)}\eta_1 + O(\frac{1}{N^2})$$

where we have:

$$\eta_1 = \frac{2^{d-3}(d-4)\Gamma(\frac{d-1}{2})sin(\frac{\pi d}{2})}{\pi^{\frac{3}{2}}\Gamma(\frac{d}{2}+1)}$$

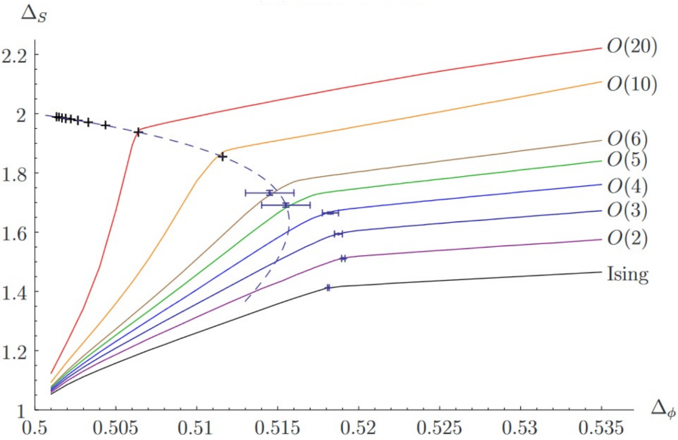
If we take $d = 4 - \epsilon$, we get exactly the same results we found using ϵ-expansion.

Case of d=3

using these method people have found:

$$\Delta_\phi = \frac{1}{2} + \frac{4}{3\pi^2}\frac{1}{N} - \frac{256}{27\pi^4}\frac{1}{N^2} + O(\frac{1}{N^3})$$
$$\Delta_s = 2 - \frac{32}{3\pi^2}\frac{1}{N} + \frac{32(16-27\pi^2)}{27\pi^4}\frac{1}{N^2} + O(\frac{1}{N^3})$$

which perfectly matches the results from Bootstrap method for kinks.


Δ_S
2.2
2
1.8
1.6
1.4
1.2
1
0.5
0.505
0.51
0.515
0.52
0.525
0.53
0.535
Δ_ϕ
$O(20)$
$O(10)$
$O(6)$
$O(5)$
$O(4)$
$O(3)$
$O(2)$
Ising

$d = 6 - \epsilon$ case

If we take $d = 6 - \epsilon$ in the results from ϵ-expansion we have:

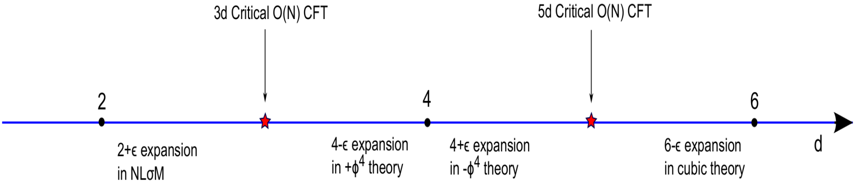
$$\Delta_\phi = 2 - \frac{\epsilon}{2} + (\frac{1}{N} + \frac{44}{N^2} + \frac{1936}{N^3} + \dots)\epsilon + O(\epsilon^2)$$
$$\Delta_\sigma = 2 + (\frac{40}{N} + \frac{6800}{N^2} + \dots)\epsilon + O(\epsilon^2)$$

Now if we consider a cubic theory with the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \phi)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2}{6}\sigma^3$$

and compute Δ_ϕ and Δ_σ for this theory, we get the same results in every order. we refer this cubic theory as UV completion theory of quartic theory.

Interacting O(N) in $2 < d < 6$



- These models are important in condensed matter physics and many body theory.
- in Holography there is a conjecture which connects these models as CFT on the boundary to the bulk:

  $$\text{Free O(N) vector model} \Leftrightarrow \text{Higher Spin Gravity in AdS}$$

  the critical O(N) model is dual to precisely the same Vasiliev theory, but with a different choice of boundary condition on the bulk scalar field.
- It has been shown that the starting point to obtain bulk dual theory is to compute Polichinski's ERG eq.

some of main references are:

- TASI Lectures on the Higher Spin - CFT duality-By Simone Giombi
- TASI Lectures on Large N Tensor Models-by Igor R. Klebanov, Fedor Popov and Grigory Tarnopolsky
- Holographic RG and Exact RG in O(N) Model-By B. Sathiapalan
- 6 Lectures on QFT, RG and SUSY-By Timothy J. Hollowood
- The Renormalization Group-By David Skinner
- Critical O(N) Models in 6 − epsilon Dimensions-By Lin Fei, Simone Giombi and Igor R. Klebanov
- Critical Properties of 4-Theories-By Hagen Kleinert and Verena Schulte-Frohlinde