Carrollian Field Theory

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PhD defense

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Based on:

Hamid Afshar, Xavier Bekaert, Mojtaba Najafizadeh
"Classification of conformal Carroll algebras"
JHEP 12 (2024) 148, arXiv:2409.19953

Along with:

Konstantinos Koutrolikos, Mojtaba Najafizadeh
"Super-Carrollian and super-Galilean field theories",
Phys. Rev. D **108**, 125014 (2023), arXiv:2309.16786

Mojtaba Najafizadeh
"Carroll-Schrödinger Equation"
Accepted in Scientific Reports (2025), arXiv:2403.11212

Outline:

Motivation

Carrollian field theory

Carroll algebra

Carrollian conformal algebra

Conformal extensions of the Carroll algebra

Infinite-dimensional conformal extensions

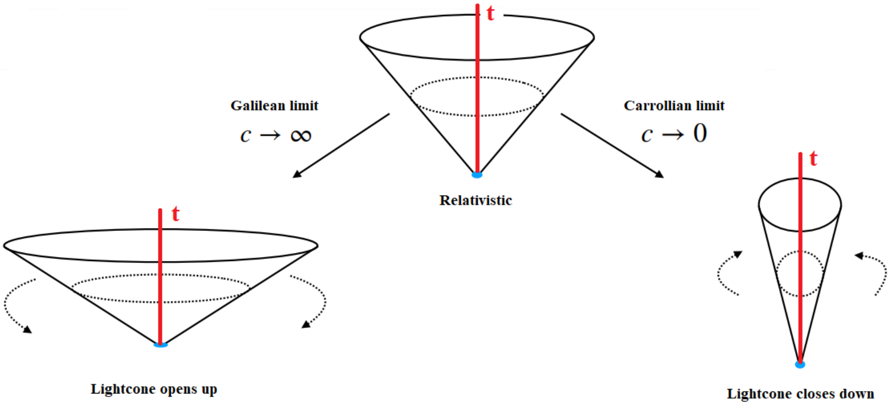
Two-point functions

Three-point functions

Outcomes

Motivation

Two limits:



Galilei limit: $\frac{v}{c} \ll 1 \quad \Longleftrightarrow \quad c \to \infty$ Carroll limit: $\frac{v}{c} \gg 1 \quad \Longleftrightarrow \quad c \to 0$

Flat space holography:
Conformal Carroll field theory might be dual to quantum gravity in asymptotically flat spacetime [S. Pasterski (2021), L. Donnay (2023), ...]

Cosmology and dark energy:
Carroll symmetry might be relevant for de Sitter cosmology and inflation [J. de Boer, J. Hartong (2022), ...]

Carroll gravity:
M. Henneaux (1979), N. A. Obers (2022), D. Grumiller (2023), ...

String theory:
Carroll symmetries arise in the tensionless limit of string theory [A. Bagchi (2016), ...]

Carroll symmetry in hydrodynamics:
L. Ciambelli, C. Marteau (2018), ...

Quantum mechanics:
M. Najafizadeh (2024), ...

Carrollian Field Theory

Let us consider the Lagrangian of a relativistic massless scalar field

$$\mathcal{L} = -\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\,\phi$$

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This Lagrangian has also an alternative by adding a Lagrange multiplier χ:

$$\mathcal{L}' = -\frac{1}{2}\,c^2\,\chi^2 + \chi\,\partial_t\phi - \frac{1}{2}\,(\partial_i\phi)^2 \qquad\qquad -\,c^2\,\chi + \partial_t\phi = 0 \tag{2}$$

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Carroll Fermions and Supersymmetry

Konstantinos Koutrolikos, Mojtaba Najafizadeh
"Super-Carrollian and super-Galilean field theories",
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Let us consider the Lagrangian density of a massless Dirac field including c

$$\mathcal{L} = -\bar{\Psi}\,(\gamma^{\mu}\partial_{\mu})\,\Psi = -\bar{\Psi}\left(\frac{1}{c}\,\gamma^{0}\partial_t + \gamma^{i}\partial_i\right)\Psi$$

Through a Carrollian limit, we found two types of Lagrangians for Carroll fermions:

electric Carroll Dirac: $\mathcal{L}_e = -\,\bar{\psi}\,\gamma^{0}\partial_t\,\psi$

magnetic Carroll Dirac: $\mathcal{L}_m = -\left(\bar{\psi}\,\gamma^{i}\partial_i\psi + \bar{\eta}\,\gamma^{0}\partial_t\,\psi + \bar{\psi}\,\gamma^{0}\partial_t\,\eta\right)$

- These Lagrangians are invariant under the Carroll boost transformations.
- Carroll fermions can be of **Dirac**, **Majorana**, or **Weyl** spinors

We introduce the **super-electric Carroll** action by:

$$\boxed{\mathrm{S}_{\mathrm{eC}}^{\mathrm{SUSY}} = \frac{1}{2}\int dt\, d^3x \left\{ (\partial_t\phi_R)^2 + (\partial_t\phi_I)^2 + {F_R}^2 + {F_I}^2 - \bar{\psi}\,\gamma^0\partial_t\,\psi \right\}}$$

and find that the action is invariant under the Super-electric Carroll transformations

$$\boxed{\begin{aligned} \delta\phi_R &= \bar{\epsilon}\,\psi \\ \delta\phi_I &= \bar{\epsilon}\,i\,\gamma^5\,\psi \\ \delta F_R &= -\,\bar{\epsilon}\,\gamma^0\,\partial_t\,\psi \\ \delta F_I &= -\,\bar{\epsilon}\,i\,\gamma^5\gamma^0\,\partial_t\,\psi \\ \delta\psi &= \gamma^0\partial_t\,(\phi_R + i\,\gamma^5\,\phi_I)\,\epsilon - (F_R + i\,\gamma^5\,F_I)\,\epsilon \end{aligned}}$$

We show that these transformations close off-shell for every field

$$[\,\delta_1\,,\,\delta_2\,] = 2\,(\bar{\epsilon}_2\,\gamma^0\,\epsilon_1)\,\partial_t$$

We introduce the **super-magnetic Carroll** action by:

$$\boxed{\mathrm{S}^{\text{SUSY}}_{\text{mC}} = \int dt\, d^3x \left\{ \chi_1\, \partial_t\, \phi_R + \chi_2\, \partial_t\, \phi_I + F_R\, G_R + F_I\, G_I + \frac{1}{2}\, \bar{\lambda}\, \gamma^0\, \psi - \frac{1}{2}\, \bar{\psi}\, \gamma^0\, \lambda \right\}}$$

and find that the action is invariant under the Super-electric Carroll transformations

$$\delta\phi_R = \bar{\epsilon}\,\psi \qquad \delta\phi_I = \bar{\epsilon}\, i\, \gamma^5\, \psi$$

$$\delta\chi_1 = \bar{\epsilon}\,\lambda \qquad \delta\chi_2 = \bar{\epsilon}\, i\, \gamma^5\, \lambda$$

$$\delta F_R = -\,\bar{\epsilon}\, \gamma^0\, \partial_t\, \psi \qquad \delta F_I = \bar{\epsilon}\, i\, \gamma^0 \gamma^5\, \partial_t\, \psi$$

$$\delta G_R = -\,\bar{\epsilon}\, \gamma^0\, \lambda \qquad \delta G_I = \bar{\epsilon}\, i\, \gamma^0 \gamma^5\, \lambda$$

$$\delta\psi = \gamma^0 \partial_t\, (\phi_R + i\, \gamma^5\, \phi_I)\, \epsilon - (F_R + i\, \gamma^5\, F_I)\, \epsilon$$

$$\delta\lambda = \gamma^0 \partial_t\, (\chi_1 + i\, \gamma^5\, \chi_2)\, \epsilon - \partial_t\, (G_R + i\, \gamma^5\, G_I)\, \epsilon$$

again these transformations close off-shell for every field

$$[\, \delta_1\, ,\, \delta_2\,] = 2\, (\bar{\epsilon}_2\, \gamma^0\, \epsilon_1)\, \partial_t$$

We also derived

Galilei fermions

and generalized them to

Supersymmetry

Carroll-Schrödinger Equation

Mojtaba Najafizadeh
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Beginning with the tachyon complex scalar field theory, we derived the **Carroll-Schrödinger** field theory through a limiting process:

In 1+1 dimensions		**In 1+d dimensions**
$\left(i\hbar\, c\, \partial_x + \frac{\hbar^2}{2mc^2}\, \partial_t^2\right)\psi = 0$		$\left(i\hbar c\, \nabla_x + \frac{\hbar^2}{2mc^2}\, \partial_t^2\right)\psi = 0$ where $\nabla_x = \frac{1}{\sqrt{x^2}}\left(x \cdot \partial + \frac{d-1}{2}\right)$
$[H, D] = H$ $[D, B] = B$ $[H, \mathbb{K}] = B$ $[H, B] = M$	$[P, D] = 2P$ $[D, \mathbb{K}] = 2\mathbb{K}$ $[P, \mathbb{K}] = D$ $[P, B] = H$	?

Canonical quantization is performed, and the two-point function is calculated

Carroll Algebra

Recall the Poincaré algebra

$$[\,P_\mu\,,\,P_\nu\,] = 0$$

$$[\,J_{\mu\nu}\,,\,P_\rho\,] = \eta_{\mu\rho}\,P_\nu - \eta_{\nu\rho}\,P_\mu$$

$$[\,J_{\mu\nu}\,,\,J_{\rho\sigma}\,] = \eta_{\mu\rho}\,J_{\nu\sigma} - \eta_{\mu\sigma}\,J_{\nu\rho} - \eta_{\nu\rho}\,J_{\mu\sigma} + \eta_{\nu\sigma}\,J_{\mu\rho}$$

in which

$$P_\mu \;:= \partial_\mu \qquad \text{Translation generators} \begin{cases} P_0 & \text{Hamiltonian} \\ P_i & \text{Momentum} \end{cases}$$

$$J_{\mu\nu} \;:= x_\mu\,\partial_\nu - x_\nu\,\partial_\mu \qquad \text{Lorentz generators} \begin{cases} J_{i0} & \text{Boosts} \\ J_{ij} & \text{Rotations} \end{cases}$$

are ten Poincaré generators in 4d spacetime.

Poincare generators, P_0, P_i, J_{i0}, J_{ij}, including c (i.e. $x^\mu = (ct, x^i)$) become:

$$\boxed{P_0 = \frac{1}{c}\,\partial_t \qquad P_i = \partial_i \qquad J_{i0} = x_i\,\frac{1}{c}\,\partial_t + c\,t\,\partial_i \qquad J_{ij} = x_i\partial_j - x_j\partial_i}$$

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Rescaling the generators, one gets

$$c\,P_0 \quad \to \quad H = \partial_t \qquad\qquad c\,J_{i0} \quad \to \quad B_i = x_i\,\partial_t + c^2\,t\,\partial_i$$

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Taking the Carroll limit $c \rightarrow 0$, one obtains the **Carroll generators:**

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satisfying the **Carroll algebra**:

$$\boxed{\begin{aligned} [P_i, B_j] &= \delta_{ij} H \\ [J_{ij}, P_k] &= \delta_{ik} P_j - \delta_{jk} P_i \\ [J_{ij}, B_k] &= \delta_{ik} B_j - \delta_{jk} B_i \\ [J_{ij}, J_{kl}] &= \delta_{ik} J_{jl} - \delta_{jk} J_{il} + \delta_{jl} J_{ik} - \delta_{il} J_{jk} \end{aligned}}$$

Carrollian Conformal Algebra

Generators of the relativistic conformal algebra, i.e. the dilatation D and special conformal transformations (SCT) K_μ, are given by:

$$D = x^\mu \partial_\mu = t\partial_t + x^i \partial_i \,, \qquad\qquad K_\mu = 2\, x_\mu\, x^\nu \partial_\nu - x^\nu x_\nu \partial_\mu \,,$$

where the critical exponent z (the coefficient of $t\partial_t$) is $z = 1$.

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satisfying the "**Carrollian conformal algebra**" (**CCA**):

$$\begin{aligned}
&[\, P_i \,,\, B_j \,] = \delta_{ij}\, H \,, && [\, D \,,\, H \,] = -\, H \,, && [\, K \,,\, P_i \,] = -\, 2\, B_i \,, \\
&[\, P_i \,,\, J_{jk} \,] = \delta_{i[j}\, P_{k]} \,, && [\, D \,,\, P_i \,] = -\, P_i \,, && [\, K_i \,,\, H \,] = -\, 2\, B_i \,, \\
&[\, B_i \,,\, J_{jk} \,] = \delta_{i[j}\, B_{k]} \,, && [\, D \,,\, K \,] = K \,, && [\, K_i \,,\, P_j \,] = -\, 2\, \delta_{ij}\, D - \, 2\, J_{ij} \,, \\
&[\, J_{ij} \,,\, J_{kl} \,] = \delta_{[i[k} J_{l]j]} \,, && [\, D \,,\, K_i \,] = K_i \,, && [\, K_i \,,\, B_j \,] = -\, \delta_{ij}\, K \,, \\
& && && [\, K_i \,,\, J_{jk} \,] = \delta_{i[j} K_{k]} \,.
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&
\end{aligned}}_{\text{Carroll algebra}}
\qquad
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\end{aligned}
\qquad
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&[\, K_i \,,\, B_j \,] = -\, \delta_{ij}\, K \,, \\
&[\, K_i \,,\, J_{jk} \,] = \delta_{i[j} K_{k]} \,.
\end{aligned}}_{\text{Conformal extension}}$$

We asked,

are there any other conformal extensions of the Carroll algebra?

Conformal extensions of the Carroll algebra

let us consider the generators $\{D, K, K_i\}$ of the Carrollian conformal algebra with some arbitrary parameters $a, b, c, \alpha, \beta, \gamma$:

$$H = \partial_t\,, \qquad P_i = \partial_i\,, \qquad B_i = x_i\,\partial_t\,, \qquad J_{ij} = x_i\,\partial_j - x_j\,\partial_i$$

$$D = a\,t\partial_t + b\,x^i\partial_i\,, \qquad K = c\,x^2\partial_t\,, \qquad K_i = 2\,\alpha\,x_i\,t\partial_t + 2\,\beta\,x_i\,x^j\partial_j - \gamma\,x^2\partial_i\,.$$

Minimal conformal extensions of the Carroll algebra

let us consider the generators $\{D, K, K_i\}$ of the Carrollian conformal algebra with some arbitrary parameters $a, b, c, \alpha, \beta, \gamma$:

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$$D = a\,t\partial_t + b\,x^i\partial_i\,, \qquad K = c\,x^2\partial_t\,, \qquad K_i = 2\,\alpha\,x_i\,t\partial_t + 2\,\beta\,x_i\,x^j\partial_j - \gamma\,x^2\partial_i\,.$$

Accordingly, one gets the following non-trivial commutation relations

$$\begin{aligned}
&[\,D\,,\,H\,] = -\,a\,H\,, && [\,K\,,\,K_i\,] = 2\,x_i\,K\,(\alpha - 2\,\beta + \gamma)\,,\\
&[\,D\,,\,P_i\,] = -\,b\,P_i\,, && [\,K_i\,,\,H\,] = -\,2\,\alpha\,B_i\,,\\
&[\,D\,,\,B_i\,] = (b-a)\,B_i\,, && [\,K_i\,,\,P_j\,] = -\,2\,\delta_{ij}\,(\alpha\,t\,\partial_t + \beta\,x^k\partial_k) - J_{ij}\,(\beta+\gamma) - \tilde{J}_{ij}\,(\beta-\gamma)\,,\\
&[\,D\,,\,K\,] = (2b-a)\,K\,, && [\,K_i\,,\,B_j\,] = -\,2\,x_i\,B_j\,(\alpha-\beta) - \,\delta_{ij}\,\gamma\,x^2\,\partial_t\,,\\
&[\,D\,,\,K_i\,] = b\,K_i\,, && [\,K_i\,,\,J_{jk}\,] = \delta_{i[j}K_{k]}\,,\\
&[\,K\,,\,P_i\,] = -\,2\,c\,B_i\,, && [\,K_i\,,\,K_j\,] = 2\,\gamma\,(\gamma-\beta)\,x^2\,J_{ij}\,,
\end{aligned}$$

where $\tilde{J}_{ij} := x_i\,\partial_j\,+\,x_j\,\partial_i\,.$

In general, one can think of the Carroll algebra as being conformally extended by adding the conformal generators (D, K, K_i) in seven different ways:

$$D, \qquad K, \qquad \cancel{K_i}, \qquad D-K, \qquad \cancel{D-K_i}, \qquad \cancel{K-K_i}, \qquad D-K-K_i\,.$$

However, one immediately finds that satisfying the closure of commutation relations cancels some of the mentioned cases. In addition, the type $D-K_i$, which was ruled out, can in fact be a possible type in $1+1$ dimensions. Therefore, the five general types of conformal Carroll algebras are:

Conformal Carroll algebras	**Symbol**	**Name**
Type K algebra	$\mathfrak{Kcarr}(d+1)$	temporal SCT-Carroll algebra
Type D algebras	$\mathfrak{scalcarr}_z(d+1)$	scaling Carroll algebras
Type D-K algebras	$\mathfrak{confcarr}_z(d+1)$	conformal Carroll algebras
Type D-K-K$_i$ algebra	$\mathfrak{cca}_1(d+1)$	Carrollian conformal algebra
Type D-K$_i$ algebra	$\mathfrak{carrsch}(1+1)$	Carroll-Schrödinger algebra

Type K algebra: $\mathfrak{Kcarr}(d+1)$

This type of algebra corresponds to the case

$$c \neq 0 \quad \text{which we set } c = 1\,, \qquad \& \qquad a = b = \alpha = \beta = \gamma = 0$$

resulting in the generators D and K_i to be trivially realised. The generators are

$$H = \partial_t\,, \qquad P_i = \partial_i\,, \qquad B_i = x_i\,\partial_t\,, \qquad J_{ij} = x_i\,\partial_j - x_j\,\partial_i\,,$$
$$K = x^2\partial_t\,,$$

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satisfying the following non-zero commutation relations

$$\begin{aligned}
&[\,P_i\,,\,B_j\,] = \delta_{ij}\,H\,, \qquad [\,K\,,\,P_i\,] = -\,2\,B_i\,,\\
&[\,P_i\,,\,J_{jk}\,] = \delta_{i[j}\,P_{k]}\,,\\
&[\,B_i\,,\,J_{jk}\,] = \delta_{i[j}\,B_{k]}\,,\\
&[\,J_{ij}\,,\,J_{kl}\,] = \delta_{[i[k}J_{l]j]}\,.
\end{aligned}$$

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&[\,J_{ij}\,,\,J_{kl}\,] = \delta_{[i[k}J_{l]j]}\,.
\end{aligned}$$

We called this the "temporal SCT-Carroll algebra" and denote it $\mathfrak{Kcarr}(d+1)$.

Type D algebras: $\mathfrak{scalcarr}_z(d+1)$

This type of algebra corresponds to the case

$$a \; or \; b \neq 0 \qquad \& \qquad c = \alpha = \beta = \gamma = 0$$

resulting in the generators K and K_i to be trivially realised. The generators are

$$H = \partial_t\,, \qquad P_i = \partial_i\,, \qquad B_i = x_i\,\partial_t\,, \qquad J_{ij} = x_i\,\partial_j - x_j\,\partial_i$$

$$D = a\,t\,\partial_t + b\,x^i\partial_i\,,$$

Type D algebras: $\mathfrak{scalcarr}_z(d+1)$

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$$H = \partial_t \,, \qquad P_i = \partial_i \,, \qquad B_i = x_i \, \partial_t \,, \qquad J_{ij} = x_i \, \partial_j - x_j \, \partial_i$$

$$D = a \, t \, \partial_t + b \, x^i \partial_i \,,$$

satisfying the following non-zero commutation relations

$$\begin{aligned}
&[\, P_i \,,\, B_j \,] = \delta_{ij} \, H \,, && [\, D \,,\, H \,] = -\, a \, H \,, \\
&[\, P_i \,,\, J_{jk} \,] = \delta_{i[j} \, P_{k]} \,, && [\, D \,,\, P_i \,] = -\, b \, P_i \,, \\
&[\, B_i \,,\, J_{jk} \,] = \delta_{i[j} \, B_{k]} \,, && [\, D \,,\, B_i \,] = (b - a) \, B_i \,, \\
&[\, J_{ij} \,,\, J_{kl} \,] = \delta_{[i[k} J_{l]j]} \,,
\end{aligned}$$

Type D algebras: $\mathfrak{scalcarr}_z(d+1)$

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$$\begin{aligned}
[\, P_i \,,\, B_j \,] &= \delta_{ij} \, H \,, & [\, D \,,\, H \,] &= -\, a \, H \,, \\
[\, P_i \,,\, J_{jk} \,] &= \delta_{i[j} \, P_{k]} \,, & [\, D \,,\, P_i \,] &= -\, b \, P_i \,, \\
[\, B_i \,,\, J_{jk} \,] &= \delta_{i[j} \, B_{k]} \,, & [\, D \,,\, B_i \,] &= (b-a) \, B_i \,, \\
[\, J_{ij} \,,\, J_{kl} \,] &= \delta_{[i[k} J_{l]j]} \,,
\end{aligned}$$

Setting $a = z$ and $b = 1$, one has $\qquad D = z \, t \, \partial_t + x^i \partial_i \,, \qquad z \in \mathbb{R}$

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&[\,P_i\,,\,B_j\,] = \delta_{ij}\,H\,, && [\,D\,,\,H\,] = -\,a\,H\,,\\
&[\,P_i\,,\,J_{jk}\,] = \delta_{i[j}\,P_{k]}\,, && [\,D\,,\,P_i\,] = -\,b\,P_i\,,\\
&[\,B_i\,,\,J_{jk}\,] = \delta_{i[j}\,B_{k]}\,, && [\,D\,,\,B_i\,] = (b-a)\,B_i\,,\\
&[\,J_{ij}\,,\,J_{kl}\,] = \delta_{[i[k}J_{l]j]}\,,
\end{aligned}$$

Setting $a = z$ and $b = 1$, one has $\qquad D = z\,t\,\partial_t + x^i\partial_i\,, \qquad z \in \mathbb{R}$

We called this the "scaling Carroll algebras" and denote it $\mathfrak{scalcarr}_z(d+1)$.

$$\begin{aligned}
\mathfrak{scalcarr}_0(d+1) &\longrightarrow \text{ spatial scaling Carroll algebra}\\
\mathfrak{scalcarr}_1(d+1) &\longrightarrow \text{ isotropic scaling Carroll algebra}\\
\mathfrak{scalcarr}_\infty(d+1) &\longrightarrow \text{ temporal scaling Carroll algebra}
\end{aligned}$$

Type D-K algebras: $\mathfrak{confcarr}_z(d+1)$

This type of algebra corresponds to the case

$$a \text{ or } b \neq 0 \; (a = z, b = 1) \qquad \& \qquad c \neq 0 \; (c = 1) \qquad \& \qquad \alpha = \beta = \gamma = 0$$

resulting in the generator K_i to be trivially realised. The generators are

$$H = \partial_t \,, \qquad P_i = \partial_i \,, \qquad B_i = x_i \, \partial_t \,, \qquad J_{ij} = x_i \, \partial_j - x_j \, \partial_i$$

$$D = z \, t \, \partial_t + x^i \partial_i \,, \qquad K = x^2 \partial_t \,,$$

This type of algebra corresponds to the case

$$a\ or\ b \neq 0\ (a = z, b = 1) \qquad \& \qquad c \neq 0\ (c = 1) \qquad \& \qquad \alpha = \beta = \gamma = 0$$

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$$D = z\,t\,\partial_t + x^i\partial_i\,, \qquad K = x^2\partial_t\,,$$

satisfying the following non-zero commutation relations

$$\begin{aligned}
&[\,P_i\,,\,B_j\,] = \delta_{ij}\,H\,, && [\,D\,,\,H\,] = -\,z\,H\,, && [\,K\,,\,P_i\,] = -\,2\,B_i\,,\\
&[\,P_i\,,\,J_{jk}\,] = \delta_{i[j}\,P_{k]}\,, && [\,D\,,\,P_i\,] = -\,P_i\,, && \\
&[\,B_i\,,\,J_{jk}\,] = \delta_{i[j}\,B_{k]}\,, && [\,D\,,\,B_i\,] = (1-z)\,B_i\,, && \\
&[\,J_{ij}\,,\,J_{kl}\,] = \delta_{[i[k}J_{l]j]}\,, && [\,D\,,\,K\,] = (2-z)\,K\,, &&
\end{aligned}$$

This type of algebra corresponds to the case

$$a \text{ or } b \neq 0 \ (a = z, b = 1) \qquad \& \qquad c \neq 0 \ (c = 1) \qquad \& \qquad \alpha = \beta = \gamma = 0$$

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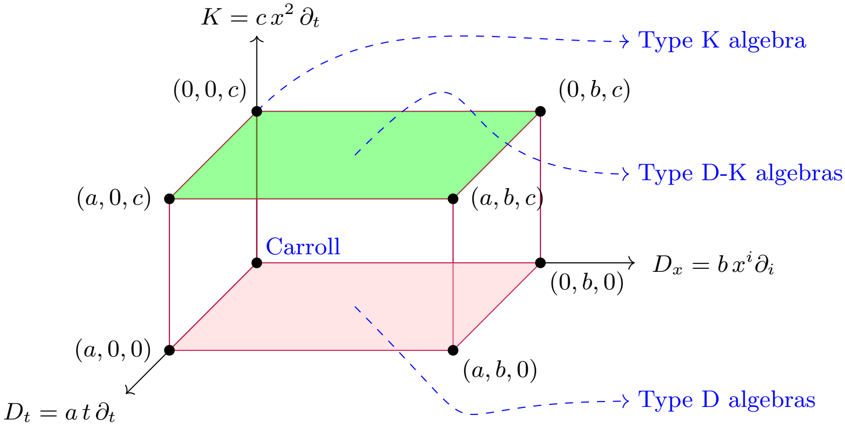
&[\,P_i\,,\,J_{jk}\,] = \delta_{i[j}\,P_{k]}\,, && [\,D\,,\,P_i\,] = -\,P_i\,, &&\\
&[\,B_i\,,\,J_{jk}\,] = \delta_{i[j}\,B_{k]}\,, && [\,D\,,\,B_i\,] = (1-z)\,B_i\,, &&\\
&[\,J_{ij}\,,\,J_{kl}\,] = \delta_{[i[k}J_{l]j]}\,, && [\,D\,,\,K\,] = (2-z)\,K\,, &&
\end{aligned}$$

We called this the "conformal Carroll algebras of dynamical exponent z" and denote it $\mathfrak{confcarr}_z(d+1)$.

These three types of algebras can be visualized in the following figure:



Type D-K-K$_i$ algebra: $\mathfrak{cca}_1(d+1)$

This type of algebra corresponds to the case

$$a = b = c = \alpha = \beta = \gamma = 1$$

The generators are

$$H = \partial_t\,, \qquad P_i = \partial_i\,, \qquad B_i = x_i\,\partial_t\,, \qquad J_{ij} = x_i\,\partial_j - x_j\,\partial_i\,,$$

$$D = t\partial_t + x^i\partial_i\,, \qquad K = x^2\partial_t\,, \qquad K_i = 2\,x_i\,(t\partial_t + x^j\,\partial_j) - x^2\partial_i\,,$$

Type D-K-K_i algebra: $\mathfrak{cca}_1(d+1)$

This type of algebra corresponds to the case

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The generators are

$$H = \partial_t\,, \qquad P_i = \partial_i\,, \qquad B_i = x_i\,\partial_t\,, \qquad J_{ij} = x_i\,\partial_j - x_j\,\partial_i\,,$$
$$D = t\partial_t + x^i\partial_i\,, \qquad K = x^2\partial_t\,, \qquad K_i = 2\,x_i\,(t\partial_t + x^j\,\partial_j) - x^2\partial_i\,,$$

satisfying the following non-zero commutation relations

$$\begin{aligned}
&[\,P_i\,,\,B_j\,] = \delta_{ij}\,H\,, && [\,D\,,\,H\,] = -\,H\,, && [\,K\,,\,P_i\,] = -\,2\,B_i\,,\\
&[\,P_i\,,\,J_{jk}\,] = \delta_{i[j}\,P_{k]}\,, && [\,D\,,\,P_i\,] = -\,P_i\,, && [\,K_i\,,\,H\,] = -\,2\,B_i\,,\\
&[\,B_i\,,\,J_{jk}\,] = \delta_{i[j}\,B_{k]}\,, && [\,D\,,\,K\,] = K\,, && [\,K_i\,,\,P_j\,] = -\,2\,\delta_{ij}\,D - \,2\,J_{ij}\,,\\
&[\,J_{ij}\,,\,J_{kl}\,] = \delta_{[i[k}J_{l]j]}\,, && [\,D\,,\,K_i\,] = K_i\,, && [\,K_i\,,\,B_j\,] = -\,\delta_{ij}\,K\,,\\
& && && [\,K_i\,,\,J_{jk}\,] = \delta_{i[j}K_{k]}\,.
\end{aligned}$$

Type D-K-K$_i$ algebra: $\mathfrak{cca}_1(d+1)$

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$$a = b = c = \alpha = \beta = \gamma = 1$$

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$$H = \partial_t \,, \qquad P_i = \partial_i \,, \qquad B_i = x_i \, \partial_t \,, \qquad J_{ij} = x_i \, \partial_j - x_j \, \partial_i \,,$$
$$D = t\partial_t + x^i \partial_i \,, \qquad K = x^2 \partial_t \,, \qquad K_i = 2 \, x_i \, (t\partial_t + x^j \, \partial_j) - \, x^2 \partial_i \,,$$

satisfying the following non-zero commutation relations

$$\begin{aligned}
&[\, P_i \,,\, B_j \,] = \delta_{ij} \, H \,, && [\, D \,,\, H \,] = - \, H \,, && [\, K \,,\, P_i \,] = - \, 2 \, B_i \,, \\
&[\, P_i \,,\, J_{jk} \,] = \delta_{i[j} \, P_{k]} \,, && [\, D \,,\, P_i \,] = - \, P_i \,, && [\, K_i \,,\, H \,] = - \, 2 \, B_i \,, \\
&[\, B_i \,,\, J_{jk} \,] = \delta_{i[j} \, B_{k]} \,, && [\, D \,,\, K \,] = K \,, && [\, K_i \,,\, P_j \,] = - \, 2 \, \delta_{ij} \, D - \, 2 \, J_{ij} \,, \\
&[\, J_{ij} \,,\, J_{kl} \,] = \delta_{[i[k} J_{l]j]} \,, && [\, D \,,\, K_i \,] = K_i \,, && [\, K_i \,,\, B_j \,] = - \, \delta_{ij} \, K \,, \\
& && && [\, K_i \,,\, J_{jk} \,] = \delta_{i[j} K_{k]} \,.
\end{aligned}$$

So we recovered the “Carrollian conformal algebra” (CCA) and denote it $\mathfrak{cca}_1(d+1)$, since $z = 1$.

Type D-K$_i$ algebra: $\mathfrak{carrsch}(1+1)$

In $1+1$ spacetime dimensions, the rotation generator J_{ij} vanishes, so another type of algebra is possible corresponding to the case

$$a = 1 \qquad b = 2 \qquad c = 0 \qquad \alpha = \frac{1}{2} \qquad \beta = \gamma = 1$$

The generators are

$$H = \partial_t\,, \qquad P = \partial_x\,, \qquad B = x\,\partial_t\,,$$
$$D = t\,\partial_t + 2\,x\,\partial_x\,, \qquad \mathbb{K} = x\,t\partial_t + x^2\,\partial_x\,,$$

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The generators are

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$$D=t\,\partial_t+2\,x\,\partial_x\,, \qquad \mathbb{K}=x\,t\partial_t+x^2\,\partial_x\,,$$

satisfying the following commutation relations

$$\begin{aligned}
&[\,D\,,\,H\,]=-\,H\,, && [\,P\,,\,B\,]=H\,,\\
&[\,D\,,\,P\,]=-\,2\,P\,, && [\,\mathbb{K}\,,\,H\,]=-\,B\,,\\
&[\,D\,,\,B\,]=B\,, && [\,\mathbb{K}\,,\,P\,]=-\,D\,,\\
&[\,D\,,\,\mathbb{K}\,]=2\,\mathbb{K}\,.
\end{aligned}$$

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satisfying the following commutation relations

$$\begin{aligned} &[\,D\,,\,H\,] = -\,H\,, && [\,P\,,\,B\,] = H\,,\\ &[\,D\,,\,P\,] = -\,2\,P\,, && [\,\mathbb{K}\,,\,H\,] = -\,B\,,\\ &[\,D\,,\,B\,] = B\,, && [\,\mathbb{K}\,,\,P\,] = -\,D\,,\\ &[\,D\,,\,\mathbb{K}\,] = 2\,\mathbb{K}\,. \end{aligned}$$

This algebra can admit a central charge M such that $[\,H\,,\,B\,] = M$, which corresponds to the Carroll-Schrödinger algebra (MN: 2403.11212), denoted by $\mathfrak{carrsch}(1+1)$.

Infinite-dimensional conformal extensions of the Carroll algebra

We find infinite-dimensional conformal extensions of the Carroll algebra, in two, three and higher spacetime dimensions, denoted by $\widetilde{\mathfrak{confcarr}}_z(d+1)$:

$$\text{Spatial conformal Carroll algebra } (z=0): \begin{cases} \widetilde{\mathfrak{confcarr}}_0(1+1) \\ \widetilde{\mathfrak{confcarr}}_0(2+1) \\ \widetilde{\mathfrak{confcarr}}_0(d+1) \end{cases}$$

$$\text{Conformal Carroll algebras } (0<|z|<\infty): \begin{cases} \widetilde{\mathfrak{confcarr}}_z(1+1) & \checkmark \\ \widetilde{\mathfrak{confcarr}}_z(2+1) & \checkmark \\ \widetilde{\mathfrak{confcarr}}_z(d+1) \end{cases}$$

$$\text{Temporal conformal Carroll algebra } (|z|=\infty): \begin{cases} \widetilde{\mathfrak{confcarr}}_\infty(1+1) \\ \widetilde{\mathfrak{confcarr}}_\infty(2+1) \\ \widetilde{\mathfrak{confcarr}}_\infty(d+1) \end{cases}$$

In 1+1 dimensions, $J_{ij} = 0$, and the generators of $\mathfrak{confcarr}_z(1+1)$ can be identified as the following generators

$$
\begin{aligned}
H &= \partial_t \ = M_0 \ , & P &= \partial_x \ = -\, L_{-1} \ , \\
B &= x\,\partial_t \ = M_1 \ , & D &= z\,t\,\partial_t + x\,\partial_x \ = -\, L_0 \ , \\
K &= x^2\partial_t \ = M_2 \ .
\end{aligned}
$$

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$$K = x^2 \partial_t \;\; = M_2 \,.$$

These generators can be extended to

$$\boxed{L_n = -\,z\,(n+1)\,x^n\,t\,\partial_t \,-\, x^{n+1}\,\partial_x \,, \qquad\qquad M_r = x^{\,r}\,\partial_t \,.}$$

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$$

satisfying

$$
\boxed{
\begin{aligned}
&[\,L_n\,,\,L_m\,] = (n-m)\,L_{n+m}\,,\\
&[\,L_n\,,\,M_r\,] = \big((n+1)\,z - r\big)\,M_{n+r}\,, \qquad (n,m,r,s \in \mathbb{Z})\\
&[\,M_r\,,\,M_s\,] = 0\,,
\end{aligned}
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$$K = x^2\partial_t \ = M_2 \ .$$

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We refer to this as $\widetilde{\mathfrak{confcarr}}_z(1+1)$, so

$$\mathfrak{confcarr}_z(1+1) \ \longrightarrow\ \widetilde{\mathfrak{confcarr}}_z(1+1)$$

The spatial SCT generator can be identified as

$$\mathbb{K} = 2z\,x\,t\partial_t + x^2\partial_x = -L_1$$

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$$\left[\, L_1\, ,\, M_r \,\right] = \big(2z - r\big)\, M_{r+1}$$

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shows that, for any integer or half-integer dynamical exponent $z = \frac{N}{2}$ with $N \in \mathbb{N}$, one can add the spatial SCT generator and truncate the supertranslation generators M_n to the finite collection with $0 \leqslant n \leqslant N$ since $[\, L_1 \,,\, M_N \,] = 0$. The Lie algebra spans by

$$\{L_{-1}, L_0, L_1, M_0, M_1, \cdots, M_N\}$$

denoted by $\mathfrak{cca}_{N/2}(1+1)$ and called the “extended Carrollian conformal algebra”.

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denoted by $\mathfrak{cca}_{N/2}(1+1)$ and called the "extended Carrollian conformal algebra".

- For $N = 1$: $\mathfrak{cca}_{1/2}(1+1) \cong \mathfrak{carrsch}(1+1)$ with generators $\{L_{-1}, L_0, L_1, \underbrace{M_0, M_1}_{Y_{\pm 1/2}}\}$

The spatial SCT generator can be identified as

$$\mathbb{K} = 2z\, x\, t\partial_t + x^2\partial_x = -L_1$$

Then the commutation relation

$$[\, L_1 \,,\, M_r \,] = \big(2z - r\big)\, M_{r+1}$$

shows that, for any integer or half-integer dynamical exponent $z = \frac{N}{2}$ with $N \in \mathbb{N}$, one can add the spatial SCT generator and truncate the supertranslation generators M_n to the finite collection with $0 \leqslant n \leqslant N$ since $[\, L_1 \,,\, M_N \,] = 0$. The Lie algebra spans by

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denoted by $\mathfrak{cca}_{N/2}(1+1)$ and called the "extended Carrollian conformal algebra".

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- For $N = 2$: $\mathfrak{cca}_1(1+1) \cong \mathfrak{bms}_3$ with generators $\{L_{-1}, L_0, L_1, \underbrace{M_0, M_1, M_2}_{M_{\pm 1,0}}\}$

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$$N = 3 : \mathfrak{cca}_{3/2}(1+1) \quad \rightarrow \quad Y_{\pm\frac{1}{2}, \pm\frac{3}{2}}$$

$$N = 4 : \mathfrak{cca}_2(1+1) \quad \rightarrow \quad M_{\pm 2, \pm 1, 0}$$

In 2+1 spacetime dimensions, we find ($w = x_1 + i\,x_2$ and $\bar{w} = x_1 - i\,x_2$)

$\widetilde{\mathfrak{confcarr}}_z(2+1)$	**generators** $(n, r, s \in \mathbb{Z})$
$[\,L_n\,,\,L_m\,] = (n-m)\,L_{n+m}$	$L_n := -\,\frac{z}{2}\,(n+1)\,w^{\,n}\,t\,\partial_t - w^{\,n+1}\,\partial_w$
$[\,\bar{L}_n\,,\,\bar{L}_m\,] = (n-m)\,\bar{L}_{n+m}$	$\bar{L}_n := -\,\frac{z}{2}\,(n+1)\,\bar{w}^{\,n}\,t\,\partial_t - \bar{w}^{\,n+1}\,\partial_{\bar{w}}$
$[\,L_n\,,\,M_{(r,s)}\,] = \left(\,\frac{z}{2}(n+1) - r\right)M_{(r+n,s)}$	$M_{(r,s)} := w^r\,\bar{w}^s\,\partial_t$
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On the gravity side, it is interesting to establish consistent asymptotically locally flat boundary conditions for general z.

Two-point functions

If we consider the two-point function

$$G^{(2)}(\vec{x}_1, t_1; \vec{x}_2, t_2) = \langle 0|\phi_1(\vec{x}_1, t_1)\, \phi_2(\vec{x}_2, t_2)|0\rangle$$

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we have

$$(\partial_{t_1} + \partial_{t_2})\, G^{(2)}(\vec{x}_1, t_1; \vec{x}_2, t_2) = 0\,, \qquad H = \partial_t$$

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One gets, $G^{(2)} = G^{(2)}(\vec{x}_{12}, t_{12})$ where $\vec{x}_{12} = \vec{x}_1 - \vec{x}_2$ and $t_{12} = t_1 - t_2$. Since $\partial_{t_1} F(t_{12}) = \partial_{t_{12}} F(t_{12})$ and $\partial_{t_2} F(t_{12}) = -\,\partial_{t_{12}} F(t_{12})$ for any function F

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which is solved as (considering $x\,\delta(x) = 0$)

$$\boxed{G^{(2)}(\vec{x}_{12}, t_{12}) = G\big(\vec{x}_{12}\big) + F(t_{12})\,\delta(\vec{x}_{12})}$$

The invariance under temporal SCT, $K = \vec{x} \cdot \vec{x}\, \partial_t$, is automatically satisfied.

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The invariance under dilatation, D, results in:

Dilatation invariance: $D = z\, t\partial_t + \vec{x} \cdot \vec{\partial}_x + \Delta$	
$G^{(2)}(\vec{x}_{12}, t_{12}) = \begin{cases} C_1\, \lvert\vec{x}_{12}\rvert^{-(\Delta_1+\Delta_2)}, & \Delta_1 + \Delta_2 \neq d \\ C_1\, \lvert\vec{x}_{12}\rvert^{-d} + F(t_{12})\delta(\vec{x}_{12}), & \Delta_1 + \Delta_2 = d \end{cases}$	$z = 0$
$G^{(2)}(\vec{x}_{12}, t_{12}) = C_1\, \lvert\vec{x}_{12}\rvert^{-(\Delta_1+\Delta_2)} + C_2\, \lvert t_{12}\rvert^{-(\Delta_1+\Delta_2-d)/z} \delta(\vec{x}_{12})$	$z \neq \{0, \infty\}$
$G^{(2)}(\vec{x}_{12}, t_{12}) = \begin{cases} C_2\, \lvert t_{12}\rvert^{-(\Delta_1+\Delta_2)}\delta(\vec{x}_{12}), & \Delta_1 + \Delta_2 \neq 0 \\ G(\vec{x}_{12}) + C_2\, \delta(\vec{x}_{12}), & \Delta_1 + \Delta_2 = 0 \end{cases}$	$z = \infty$

The invariance under spatial SCT, K_i results in:

$\mathbf{K}_i$ **invariance:** $K_i = 2\,x_i\,(z\,t\partial_t + x^j\partial_j + \Delta) - x^2\partial_i$	
$G^{(2)}(\vec{x}_{12}, t_{12}) = C_1\,\lvert\vec{x}_{12}\rvert^{-2\Delta} + C_2\,\lvert t_{12}\rvert^{\frac{d-2\Delta}{z}}\delta(\vec{x}_{12})$	$\Delta_1 = \Delta_2 = \Delta$
$G^{(2)}(\vec{x}_{12}, t_{12}) = C_2\,\lvert t_{12}\rvert^{\frac{d-(\Delta_1+\Delta_2)}{z}}\delta(\vec{x}_{12})$	$\Delta_1 \neq \Delta_2$

Three-point functions

If we consider the three-point function

$$G^{(3)}(\vec{x}_1, \vec{x}_2, \vec{x}_3; t_1, t_2, t_3) := \langle 0|\phi_1(t_1, \vec{x}_1)\phi_2(t_2, \vec{x}_2)\phi_3(t_3, \vec{x}_3)|0\rangle$$

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Invariance under the Carrollian boost $\vec{B} = \vec{x}\,\partial_t$ results in

$$\begin{aligned}(\vec{x}_1\partial_{t_1} + \vec{x}_2\partial_{t_2} + \vec{x}_3\partial_{t_3})G^{(3)}(\vec{x}_{12}, \vec{x}_{23}, \vec{x}_{31}; t_{12}, t_{23}, t_{31}) &= 0\\ (\vec{x}_{12}\partial_{t_{12}} + \vec{x}_{23}\partial_{t_{23}})G^{(3)}(\vec{x}_{12}, \vec{x}_{23}, \vec{x}_{31}; t_{12}, t_{23}, t_{31}) &= 0\,,\end{aligned}$$

If we consider the three-point function

$$G^{(3)}(\vec{x}_1, \vec{x}_2, \vec{x}_3; t_1, t_2, t_3) := \langle 0|\phi_1(t_1, \vec{x}_1)\phi_2(t_2, \vec{x}_2)\phi_3(t_3, \vec{x}_3)|0\rangle$$

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which is solved as,

$$\boxed{\begin{aligned}G^{(3)}(\vec{x}_{12}, \vec{x}_{23}, \vec{x}_{31}; t_{12}, t_{23}, t_{31}) &= G\big(|\vec{x}_{12}|, |\vec{x}_{23}|\big) + F(t_{12}, t_{23})\,\delta(\vec{x}_{12})\,\delta(\vec{x}_{23})\\ &+ F_1\big(|\vec{x}_{12}|; t_{23}\big)\,\delta(\vec{x}_{23}) + F_2\big(|\vec{x}_{23}|; t_{31}\big)\,\delta(\vec{x}_{31}) + F_3\big(|\vec{x}_{31}|; t_{12}\big)\,\delta(\vec{x}_{12})\\ &+ \mathcal{F}_1\left(|\vec{x}_{12}|, |\vec{x}_{23}|\,;\,\frac{t_{12}}{|\vec{x}_{12}|} - \frac{t_{23}}{|\vec{x}_{23}|}\right)\,\delta\left(\frac{\vec{x}_{12}}{|\vec{x}_{12}|} - \frac{\vec{x}_{23}}{|\vec{x}_{23}|}\right)\\ &+ \mathcal{F}_2\left(|\vec{x}_{23}|, |\vec{x}_{31}|\,;\,\frac{t_{23}}{|\vec{x}_{23}|} - \frac{t_{31}}{|\vec{x}_{31}|}\right)\,\delta\left(\frac{\vec{x}_{23}}{|\vec{x}_{23}|} - \frac{\vec{x}_{31}}{|\vec{x}_{31}|}\right)\\ &+ \mathcal{F}_3\left(|\vec{x}_{31}|, |\vec{x}_{12}|\,;\,\frac{t_{31}}{|\vec{x}_{31}|} - \frac{t_{12}}{|\vec{x}_{12}|}\right)\,\delta\left(\frac{\vec{x}_{31}}{|\vec{x}_{31}|} - \frac{\vec{x}_{12}}{|\vec{x}_{12}|}\right)\,,\end{aligned}}$$

Dilatation invariance determines $G, F, F_i, \mathcal{F}_i$, Dilatation invariance determines $G, F, F_i, \mathcal{F}_i$, and K-invariance remove collinear terms such that

$$\boxed{\begin{gathered} G^{(3)}(\vec{x}_{12}, \vec{x}_{23}, \vec{x}_{31}; t_{12}, t_{23}, t_{31}) = G\big(|\vec{x}_{12}|, |\vec{x}_{23}|\big) + F(t_{12}, t_{23})\, \delta(\vec{x}_{12})\, \delta(\vec{x}_{23}) \\ + F_1\big(|\vec{x}_{12}|; t_{23}\big)\, \delta(\vec{x}_{23}) + F_2\big(|\vec{x}_{23}|; t_{31}\big)\, \delta(\vec{x}_{31}) + F_3\big(|\vec{x}_{31}|; t_{12}\big)\, \delta(\vec{x}_{12}) \end{gathered}}$$

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K_i-invariance gives

$$\boxed{G(\vec{x}_{12},\vec{x}_{23},\vec{x}_{31}) = \frac{C_1}{|\vec{x}_{12}|^{\Delta_1+\Delta_2-\Delta_3}|\vec{x}_{23}|^{\Delta_3+\Delta_2-\Delta_1}|\vec{x}_{31}|^{\Delta_3+\Delta_1-\Delta_2}}}$$

and

$$\boxed{\begin{aligned} F_1\big(|\vec{x}_{12}|,t_{23}\big) &= \frac{K_1}{|\vec{x}_{12}|^{2\Delta_1}|t_{23}|^{\frac{\Delta_2+\Delta_3-\Delta_1-d}{z}}} \\ F_2\big(|\vec{x}_{23}|,t_{31}\big) &= \frac{K_2}{|\vec{x}_{23}|^{2\Delta_2}|t_{31}|^{\frac{\Delta_1+\Delta_3-\Delta_2-d}{z}}} \\ F_3\big(|\vec{x}_{31}|,t_{12}\big) &= \frac{K_3}{|\vec{x}_{31}|^{2\Delta_3}|t_{12}|^{\frac{\Delta_1+\Delta_2-\Delta_3-d}{z}}} \end{aligned}}$$

PhD Outcomes

Published papers:

Mojtaba Najafizadeh
"Massive to massless by applying a nonlocal field redefinition"
Phys. Rev. D **107**, 045008 (2023), arXiv:2212.07042

Konstantinos Koutrolikos, Mojtaba Najafizadeh
"Super-Carrollian and super-Galilean field theories",
Phys. Rev. D **108**, 125014 (2023), arXiv:2309.16786

Hamid Afshar, Xavier Bekaert, Mojtaba Najafizadeh
"Classification of conformal Carroll algebras"
JHEP 12 (2024) 148, arXiv:2409.19953

Mojtaba Najafizadeh
"Carroll-Schrödinger Equation"
Accepted in Scientific Reports (2025), arXiv:2403.11212

PhD outcomes:

Presentations:

- **International:**

  Classification of Conformal Carroll Algebras, Poster presentation at Strings 2025 conference, New York University, Abu Dhabi, UAE (Jan. 2025).

  Off-shell supersymmetric continuous spin gauge theory, Center for Theoretical Physics, Tomsk State Pedagogical University, Tomsk, Russia (July 2023) [**online**].

- **National:**

  Carroll Schrödinger Equation, Department of Physics, Ferdowsi University of Mashhad, Mashhad, Iran (Feb 2025).

  Carroll Fermions and Supersymmetry, School of Particles and Accelerators, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran (Jan 2024).

  Massive to massless by applying a nonlocal field redefinition, 30^{th} IPM Physics Spring Conference, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran (May 2023).

  Massive to massless by applying a nonlocal field redefinition, School of Physics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran (Feb. 2023).

Thank you for your attention!