Short review on critical O(N) vector models

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Preface

This is a homework assignment for the quantum field theory III course taught by Prof. H. Afshar in the spring semester of 1402. In this assignment, I briefly explored various aspects of critical QFTs, especially the fascinating model of critical O(N) vector model. I aimed to make this work as clear as possible and mentioned my main sources for further reading for those who want to learn more. Finally, I would like to express my gratitude to Dr. Afshar for his excellent course, Mr. Mehdi Ameri and Dr. Bahram Shakerin for their helpful discussions.

Chapter 1

Review on basic concepts

In this chapter we want to give a general introduction on some necessary basic concepts, that we need to know to study critical behavior of $O(N)$ vector models.

1.1 Effective action

Here we want to deal with theories From the point of view of accessible energy. At first consider the general form of real scalar field theory:

$$S_{\Lambda_0} = \int d^d x \left[\frac{1}{2} (\partial^\mu \phi \partial_\mu \phi) + \sum_i \Lambda_0^{d-d_i} g_{0i} O_i(x) \right] \tag{1.1}$$

Λ_0 is cutoff (a limit on high-energy), $d_i = [O_i]$, d is the space-time dimension and $O_i(x)$ is any operator made of field and it's derivatives. Now we want to separate our modes in two part: High-energy modes and the low-energy ones.

$$\begin{aligned} \phi(x) &= \int_{|p| \leq \Lambda} \frac{d^d p}{(2\pi)^d} e^{ip.x} \tilde{\phi}(p) + \int_{\Lambda < |P| \leq \Lambda_0} \frac{d^d p}{(2\pi)^d} e^{ip.x} \tilde{\phi}(p) && (1.2) \\ &:= (\varphi)_L + (\chi)_H && (1.3) \end{aligned}$$

Where φ and χ denotes, low and high energy modes. Also we set $\Lambda = b\Lambda_0, \quad 0 < b < 1$.

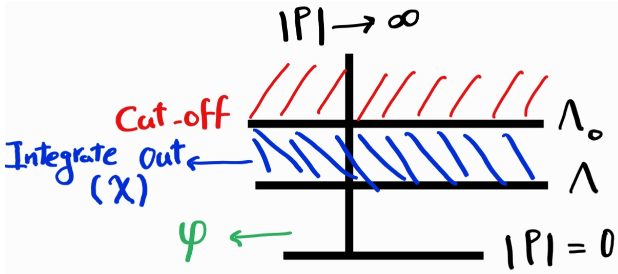

Figure 1.1: Red zone is beyond cut-off. Blue zone is where we want to integrate out. White zone is we want to keep.

Next, we can obtain the effective action by integrating out high-energy modes, we can find effective action:

$$\mathcal{Z} = \int \mathcal{D}\varphi \int \mathcal{D}\chi e^{\frac{-S_{\Lambda_0}}{\hbar}} \quad \rightarrow \quad S_\Lambda^{eff}(\varphi) := -\hbar \ln \left[\int \mathcal{D}\chi \exp(\frac{-S_{\Lambda_0}}{\hbar}) \right] \tag{1.4}$$

Also we can write:

$$S_\Lambda^{eff}(\varphi) = S^0(\Phi) + S_\Lambda^{int}(\varphi) \tag{1.5}$$

Where

$$S^0(\varphi) = \int d^d x (\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m^2\varphi^2) \tag{1.6}$$

$$S_\Lambda^{int}(\varphi) = -\ln \int \mathcal{D}\chi exp(-S^0(\chi) - S_{\Lambda_0}^{int}(\varphi + \chi)) \tag{1.7}$$

We want to ask by changing the scale of energy what happens to our system? to answer this we need to introduce two important functions. First one is β function, which tells us how the interacting coupling change when we change the scale of energy. In next parts of this note, we use it many times and get information about our theory. This function is in the form:

$$\beta_i(g) := \Lambda \frac{\partial g_i}{\partial \Lambda} \tag{1.8}$$

Before introducing next important function, it is worth to say that our initial action of scalar field will have the general form of:

$$S_\Lambda^{eff}(\varphi) = \int d^d x (\frac{Z_\Lambda}{2}(\partial\varphi)^2 + \sum_i \Lambda^{d-d_i} Z_\Lambda^{\frac{n_i}{2}} g_i(\Lambda) O_i) \tag{1.9}$$

Where Z_Λ is Renormalization factor: $\varphi^{nor} := Z_\Lambda^{\frac{1}{2}}\varphi$. Now we introduce anomalous dimension as:

$$\gamma_\phi := \frac{-1}{2}\Lambda\frac{\partial \ln Z_\Lambda}{\partial \Lambda} \tag{1.10}$$

This is something like β function but for renormalization factors. Using anomalous dimension one can define scaling dimension:

$$\Delta_\phi \equiv (\frac{d-2}{2}) + \gamma_\phi$$

If $\gamma_\phi = 0$, this is the same mass dimension of field. In other words, the anomalous dimension is how much the scaling dimension deviates from the expected classical dimension.

For the n-point function, we obtain:

$$\langle\varphi(x_1)\ldots\varphi(x_n)\rangle = Z_\Lambda^{\frac{-n}{2}}\langle\varphi^{nor}(x_1)\ldots\varphi^{nor}(x_n)\rangle \equiv Z_\Lambda^{\frac{-n}{2}}\Gamma_\Lambda^{(n)}(x_1,\ldots,x_n;g_i(\Lambda))$$

Actually we can go deep in to lower energies using parameter $s < 1$ (if all modes have energies $<< \Lambda$) the n-point function will not change:

$$Z_{s\Lambda}^{\frac{-n}{2}}\Gamma_{s\Lambda}^{(n)}(x_1,\ldots,x_n;g_i(s\Lambda)) = Z_\Lambda^{\frac{-n}{2}}\Gamma_\Lambda^{(n)}(x_1,\ldots,x_n;g_i(\Lambda))$$

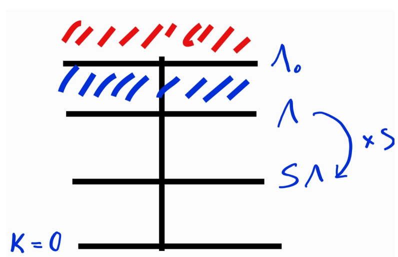

Figure 1.2: Integrating out new area.

Following this idea we can have Callan–Symanzik equation for n-point function:

$$\Lambda\frac{d\Gamma_\Lambda^{(n)}}{d\Lambda} = (\Lambda\frac{\partial}{\partial\Lambda} + \beta_i\frac{\partial}{\partial g_i} + n\gamma_\phi)\Gamma_\Lambda^{(n)} = 0 \tag{1.11}$$

This equation shows that all functions change in a manner which Γ remains unchanged.

1.2 RG flow and RG trajectory

From a historical point of view the technique we use in RG flow in its modern language came from condensed matter physics by the work of Leo Kadanoff's paper in 1966 and later by work of Kenneth G. Wilson.

we want to see what happens during renormalization? first we need to know what we get if β function become zero. The answer is we have critical-point, so our theory in that point is critical. The easiest solution to β function is the Gaussian critical-point:

$$\text{if} \quad \beta = 0 \quad \rightarrow \quad g_i = g_i^* \tag{1.12}$$

In the Gaussian critical-point we have free theory. We also call this critical-point trivial.Using Callan-Symanzik equation for two-point function, one can show that in critical-point two point functions have following behavior:

$$\Gamma_{\Lambda}^{(2)} \sim \frac{c(g_i^*)}{|x-y|^{2\Delta_\phi}} \tag{1.13}$$

This behavior is characteristic of scale invariant theories. Actually one can prove firmly that theories at critical-points have a bigger symmetry group, named conformal group. So our quantum field theories at critical-points are conformal field theories (this theorem is valid for $d = 2$, and we expect it to be true for $d > 2$). The conformal group is the group which keeps the angle between the lines invariant:

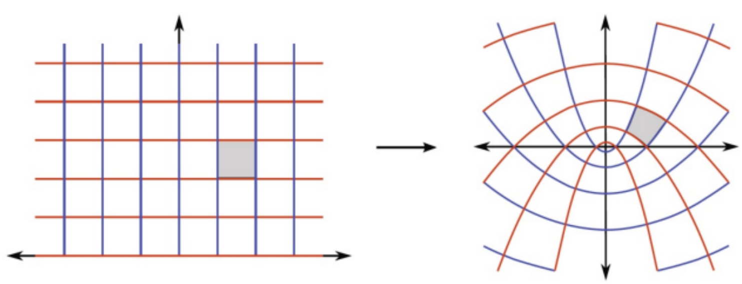

Figure 1.3: Conformal map which preserves the angel between lines.

If we want to have the shortest definition of conformal group we say it is made of Poincare group, Dilation (scale transformation) and special conformal transformation.

Now we want to ask what is going on near the critical-points? Consider we sit just a little bit away from critical-point at $g_i = g_i^* + \delta g_i$. The running for this point is:

$$\Lambda \frac{\partial g_i}{\partial \Lambda}|_{g_i^* + \delta g_i} = \beta_{ij} \delta g_j + O(\delta g^2) \tag{1.14}$$

We call β_{ij} beta matrix. Which its eigenvalues are constant. This matrix has the eigenvalue equation as the form:

$$\beta_{ij} \sigma_j = (\Delta_i - d) \sigma_j \tag{1.15}$$

Where we considered its eigenvalue $\Delta_i - d$ and σ_i is its eigenvector. Using the last two equations one can see:

$$\sigma_i(\Lambda) = (\frac{\Lambda}{\Lambda_0})^{\Delta_i - d} \sigma_i(\Lambda_0) \tag{1.16}$$

This equation is RG flow equation for operator σ. There can be three possible situation for $\Delta_i - d$:

- If $\Delta_i > d$ we have "Irrelevant" operator. By going to IR this fades away and we "flow back" to critical-point.

- If $\Delta_i < d$ we have "Relevant" operator. This operator will survive in IR.

- If $\Delta_i = d$ we have "Marginal" operator. This operator won't change in IR.

In the first case, our irrelevant couplings live on a surface, named the critical surface. All trajectories on it can be interpreted as local coordinates of this surface. This shows that irrelevant terms have no role in IR limit so we can drop them in IR regime with no concern. On the other hand relevant terms are important in IR and will grow as we go down deep to IR scales. Marginal terms will not change by going to IR and take their initial values. You can see these conditions in the space of theory.

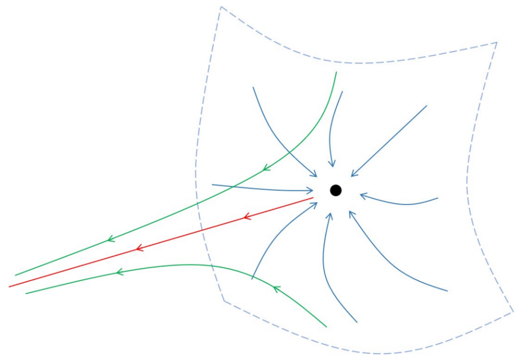

Figure 1.4: Blue lines represents irrelevant terms pointing toward critical-point. Green trajectories illustrate the relevant ones, they do not live on critical surface. Red line is coming from critical point. Note that all trajectories point towards infrared. This image has been taken from [12]

One may ask, what is the destination of all relevant trajectories? there are some possibilities, one is the continue forever. The other is the may meet each other at another critical-point in IR as we shown below:

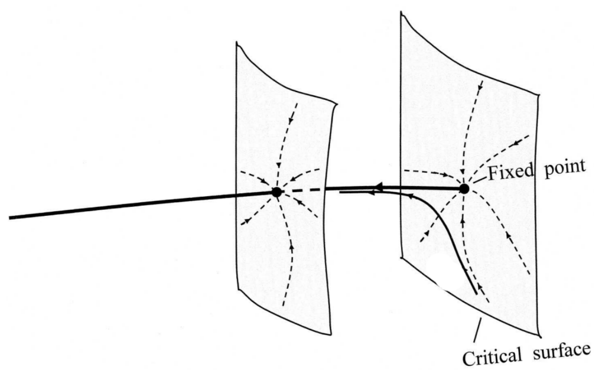

Figure 1.5: One possible destination for RG trajectories.

In UV regimes, we may have some different, theories with different interactions, but they become the same as we go to IR. We say these theories are in the same class of universality.

1.3 Polchinski's ERG equation and local potential approximation

We have defined S_{Λ}^{int} as:

$$S_{\Lambda}^{int}(\Phi) = -\ln \int \mathcal{D}\chi exp(-S^{0}(\chi) - S_{\Lambda_0}^{int}(\Phi + \chi)) \tag{1.17}$$

But working with this term directly, is a hard task. If we expand $S_{\Lambda_0}^{int}$ as following way:

$$\exp(-S_{\Lambda_0}^{int}(\varphi + \chi)) = \exp(S_{\Lambda_0}^{int}(\varphi)) + \int dx\chi(x) \frac{\delta}{\delta\varphi(x)} \exp(S_{\Lambda_0}^{int}(\varphi)) \tag{1.18}$$

$$+ \frac{1}{2} \int dxdy\chi(x)\chi(y) \frac{\delta}{\delta\varphi(x)} \frac{\delta}{\delta\varphi(y)} \exp(S_{\Lambda_0}^{int}(\varphi)) + \dots \tag{1.19}$$

For each field χ the propagator takes the following form if we impose to be in momentum shell $\Lambda' < |p| < \Lambda_0$ where $\Lambda' = \Lambda_0 - \delta\Lambda$ for small $\delta\Lambda$:

$$D_{\Lambda}^{\chi}(x, y) \sim \delta\Lambda f(\Lambda, m^2, \dots) \tag{1.20}$$

So by considering small $\delta\Lambda$, we see that each propagator of field χ has this factor and become smaller and smaller if we add new propagators. In short we need to keep only terms in our expansion, which have only two or less χ. By using this idea Joseph Polchinski found this equation named Polchinski exact renormalization group equation[14]:

$$-\Lambda \frac{\partial S_{\Lambda}^{int}(\phi)}{\partial\Lambda} = \int d^d x d^d y (\frac{\delta S_{\Lambda}^{int}(\phi)}{\delta\phi(x)} D_{\Lambda}(x, y) \frac{\delta S_{\Lambda}^{int}(\phi)}{\delta\phi(y)} - D_{\Lambda}(x, y) \frac{\delta^2 S_{\Lambda}^{int}(\phi)}{\delta\phi(x)\delta\phi(y)}) \tag{1.21}$$

This equation has exact information about the behavior of system under RG flow. For example you can check appendix c from [6] to see one of these applications in finding eigenpotential. We can also have this equation for partition function of the theory. One can rewrite this equation in the following form:

$$\frac{\partial}{\partial t} e^{-S^{int}(\phi)} = -\int d^d x d^d y (D_{\Lambda}(x, y) \frac{\delta^2}{\delta\phi(x)\delta\phi(y)} e^{-S^{int}(\phi)}) \tag{1.22}$$

Where $t \equiv \ln(\Lambda)$, we call this parameter "RG time". In AdS/CFT framework, this time is the actual time in AdS bulk and its changes show the direction of scale of energy:

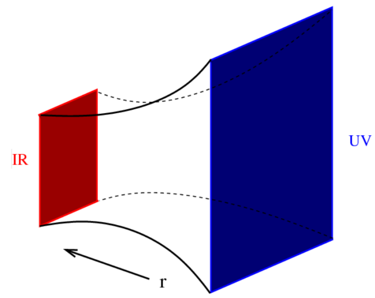

Figure 1.6: The extra direction, r, should be thought of as energy scale.

By defining the Laplacian operator we get the format of heat equation:

$$\Delta \equiv \int d^d x d^d y D_\lambda \frac{\delta^2}{\delta\phi(x)\delta\phi(y)} \qquad \rightarrow \qquad \frac{\partial}{\partial t} e^{-s_{int}} = -\Delta e^{-s_{int}} \tag{1.23}$$

Although, Polchinski's ERG equation has exact information about all function we are interested to compute, but Contrary to its simple format, solving this equation is really hard. This is our main motivation to study local potential approximation.

Consider cases with $d > 2$, if we drop derivatives of fields from our action (except for kinetic term) because they are irrelevant. We can have:

$$S_\Lambda^{eff}(\phi) = \int d^d x (\frac{1}{2}(\partial\phi)^2 + V(\phi)) \tag{1.24}$$

$$V(\phi) = \sum_K \Lambda^{d-k(d-2)} \frac{g_{2k}}{(2k)!} \phi^{2k} \tag{1.25}$$

Here we keep focus on ϕ^{2k} because we want to have Z_2 symmetry for our action[1]. We call this procedure of doping derivatives from potential, local potential approximation.

We again break our field to $\phi = \varphi + \chi$ and expand effective action as:

$$S_\Lambda^{eff}(\phi = \varphi + \chi) = S_\Lambda^{eff}(\varphi) + \int d^d x (\frac{1}{2}(\partial\chi)^2 + \chi V'(\varphi) + \frac{1}{2}\chi^2 V''(\varphi) + \dots) \tag{1.26}$$

Now if we choose to be at minimum of the potential $V'(\varphi) = 0$ we get:

$$S_\Lambda^{eff}(\phi = \varphi + \chi) = S_\Lambda^{eff}(\varphi) + \int d^d x (\frac{1}{2}(\partial\chi)^2) + \frac{1}{2}\chi^2 V''(\varphi) + \dots) \tag{1.27}$$

[1] $Z_2 : \phi \rightarrow -\phi \Rightarrow V(\phi) = V(-\phi)$

As before we again set $\Lambda' = \Lambda - \delta\Lambda$, by doing this we will see that each loop for χ field has a factor $\delta\Lambda$:

$$\int_{\Lambda'<|p|<\Lambda} \frac{d^d p}{(2\pi)^d}(f(p)) = \delta\Lambda(\dots) \tag{1.28}$$

which $f(p)$ is a function of p and by dots on the left hand side we mean all terms made of propagator and vertex. So if we choose $\delta\Lambda$ be small, this tell us we should focus of one-loop corrections. So we keep diagrams with one loop and $2k$ external field of φ:

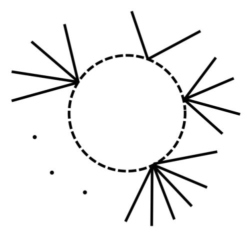

Figure 1.7: χ field one loop (dashed-line) and $2k$ external field φ.

This is equivalent to truncate terms after quadratic term in χ and only keep χ^2 terms. The result of truncation is:

$$S_\Lambda^{eff}(\phi = \varphi + \chi) - S_\Lambda^{eff}(\varphi) = S^{(2)}(\chi) = \int d^d x(\frac{1}{2}(\partial\chi)^2 + \frac{1}{2}V''(\varphi)\chi^2) \tag{1.29}$$

We can also use Fourier transformation and get Gaussian integral of the form:

$$S^{(2)}(\chi) = \int_{\Lambda'<|p|<\Lambda} \frac{d^d p}{(2\pi)^d}\chi(p)(\frac{1}{2}p^2 + \frac{1}{2}V''(\varphi))\chi(-p) \tag{1.30}$$

Now if we consider φ to be constant, so we get:

$$e^{-\delta_\Lambda S^{eff}(\varphi)} = \int \mathcal{D}\chi exp(-S^{(2)}(\varphi, \chi)) = C(\frac{\pi}{\Lambda^2 + V''(\varphi)})^{\frac{N}{2}} \tag{1.31}$$

Here C is some factor. By considering the regularization of N as the number of modes in shell of momentum, we can rewrite this expression for spatially varying φ:

$$\delta_\Lambda S^{eff}(\varphi) = a\Lambda^{d-1}\delta\Lambda \int d^d x \ln(\Lambda^2 + V''(\varphi)) \tag{1.32}$$

Here we have $a = \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}$ (see [15]). One can follow the calculation by expanding in powers of φ and get the following equation:

$$\Lambda\frac{dg_{2K}}{d\Lambda} = (k(d-2)-d)2g_{2k} - a\Lambda^{k(d-2)}\frac{\partial^{2k}}{\partial\phi^2 k}\ln(\Lambda^2 + V''(\phi))|_{\phi=0} \tag{1.33}$$

Using the last equation we can have :

$$k = 1 \rightarrow \Lambda\frac{dg_2}{d\Lambda} = -2g_2 - \frac{ag_4}{1+g_2} \qquad \text{Mass term} \tag{1.34}$$

$$k = 2 \rightarrow \Lambda\frac{dg_4}{d\Lambda} = (d-4)g_4 - \frac{ag_6}{1+g_2} + \frac{3ag_4^2}{(1+g_2)^2} \qquad \phi^4 \text{ term} \tag{1.35}$$

1.4 Wilson-Fisher critical point

In last parts we talked about the Gaussian critical-point and called them trivial, here we want to introduce an other critical-point named Wilson-Fisher critical-point. In the case of ϕ^4 scalar theory, the interaction term in cases that we are in $d < 4$ is relevant so it is clear that as we go down deep in IR area, we have strongly couple interaction. The trick here is to stay near fourth dimension $d = 4 - \epsilon$. By using this trick and imposing this to the equation we found after local potential approximation, Wilson and Fisher found a non-trivial critical-point, we denote by g_i^{WF}:

$$g_2^{WF} = -\frac{1}{6}\epsilon + O(\epsilon^2), \qquad g_4^{WF} = \frac{1}{3a}\epsilon + O(\epsilon^2) \tag{1.36}$$

And for all $k > 2$:

$$g_{2k}^{WF} \sim \epsilon^k \tag{1.37}$$

Note that these couplings show one critical-point together not two different points.

Now we want to see what is happening near this new critical-point? to find the answer, as before we linearize β function and have $\delta g_{2k} = g_{2k} - g_{2k}*$:

$$\Lambda\frac{\partial}{\partial\Lambda}\begin{pmatrix}\delta g_2\\ \delta g_4\end{pmatrix} = \begin{pmatrix}\frac{\epsilon}{3}-2 & -a(1+\frac{\epsilon}{6})\\ 0 & \epsilon\end{pmatrix} \tag{1.38}$$

We can work with the matrix and find it's eigenvalues and it's eigenvectors as:

$$\text{eigenvalues:} \quad \lambda_2 = \frac{\epsilon}{3} - 2, \qquad \lambda_4 = \epsilon \tag{1.39}$$

$$\text{eigenvectors:} \quad \sigma_2 = \begin{pmatrix}1\\ 0\end{pmatrix}, \quad \sigma_4 = \begin{pmatrix}-a(3+\frac{\epsilon}{2})\\ 2(3+\epsilon)\end{pmatrix} \tag{1.40}$$

Here we have for factor a:

$$a = \frac{1}{16\pi^2} + \frac{\epsilon(1 - \gamma_{EM} + \ln 4\pi)}{32\pi^2} + O(\epsilon^3) \tag{1.41}$$

γ_{EM} is the Euler-Mascheroni constant $\gamma_{EM} \simeq 0.5772$. About the scaling dimension one can compute easily using $\Delta - d = eigenvalue$:

$$\Delta_2 = 2 - \frac{2\epsilon}{3}, \qquad \Delta_4 = 4 \tag{1.42}$$

As we set ϵ to be small, $\lambda_2 = \frac{\epsilon}{3} - 2$ will be negative, this term is relevant near the Wilson-Fisher critical-point. The other eigenvalue will be positive, so it is irrelevant. We can see the situation of these couplings and critical-points in $d = 3$ below:

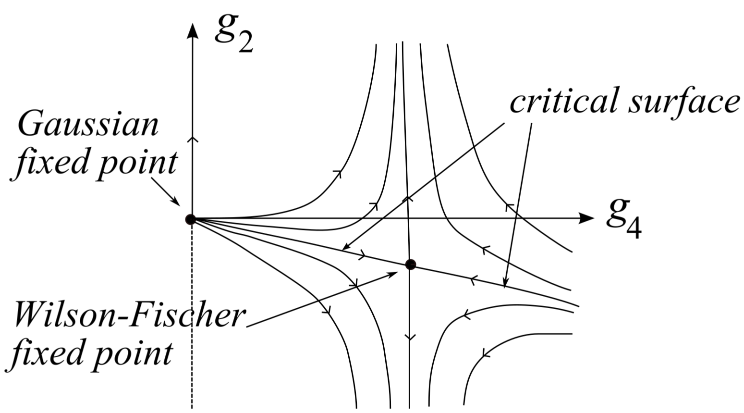

Figure 1.8: RG flow of scalar theory, spanned by g_2 and g_4 in $d = 3$. Arrow in these trajectories point in the direction of RG flow towards IR regime. This image has been taken from [5]

In these diagrams we can see two critical-points, Wilson-Fisher and Gaussian, there is trajectory from Gaussian to Wilson-Fisher which is for those theories that they are massless and free in UV but have interaction (ϕ^4) and negative mass in IR. Theories whose trajectory is vertical has no mass and no interaction in UV and id non-interacting with positive mass in IR. Those theories which do not live on the critical surface, may have positive or negative mass, in IR area.

Chapter 2

The O(N) vector model

In this part our main goal is to study a special kind of conformal field theory (field theories at critical-point) using two method:

- Wilson-Fisher ϵ-expansion.
- $\frac{1}{N}$ expansion and large N limit.

We will see that these two approach match really nice. Besides the theory of ϕ^4 interaction, which we investigate in different dimensions, at the end we will talk about a cubic theory as the UV completion in $d = 6 - \epsilon$ and going to see these two theory match at the level of scaling-dimension.

2.1 Origins from physical phenomena

O(N) Vector models are useful for various fields, such as AdS/CFT, condensed matter physics and many-body systems. In statistical systems when we are near critical point correlation length is very large, and our system becomes scale invariant so CFT's (as critical O(N) models) are good candidate to describe them. For example These O(N) vector models can be applied to investigate the O(N) ferromagnet. The energy that follows defines these systems (for $J > 0$):

$$\mathcal{E} = -J \sum_{\langle ij \rangle} \vec{n}_i \cdot \vec{n}_j, \quad \vec{n} = \left(n^1, \ldots, n^N\right), \quad \vec{n}^2 = 1 \tag{2.1}$$

This physical system has $O(N)$ symmetry. In our notation by $\langle ij \rangle$ we mean that we considered nearest neighbor. For $N = 3$ it is Heisenberg model and for $N = 1$ is Ising model. you can see a simple illustration of this system below.

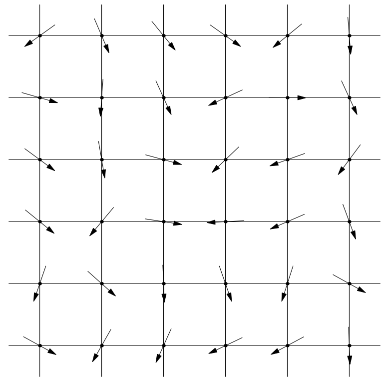

Figure 2.1: Two dimensional $O(N)$ symmetric spin system. This image has been taken from [7]

The partition function of the system is:

$$Z = \sum_{\{\vec{n}_i\}} e^{-\beta \mathcal{E}} \tag{2.2}$$

Here $\beta = \frac{1}{T}$ and T is temperature. This system has a critical temperature T_c in which the ξ (correlation length) diverges. At point near T_c :

$$\langle n^i(x) n^j(0) \rangle = \delta^{ij} \frac{1}{|x|^{2(\frac{d-2}{2})}} e^{\frac{-|x|}{\xi}} \tag{2.3}$$

When $\xi >> 1$ i.e. near critical value of β and J we can describe our system using Euclidean quantum field theory as:

$$S = \int d^d x \left[\frac{1}{2} \left(\nabla \phi_i \right)^2 + \frac{m^2}{2} \phi_i^2 + \frac{g}{4} (\phi_i^2)^2 \right] \tag{2.4}$$

Here we have used $\phi^i \equiv \langle n_i \rangle$ as an average value of n_i.

Another example is the water vapor phase diagram, This system corresponds to the $d = 3$ Ising model. Again near critical-point we can use massless real scalar field with $\lambda \phi^4$ interaction:

$$S = \int d^3 x \left(\frac{1}{2} (\partial \phi)^2 + \frac{\lambda}{4!} (\phi)^4 \right) \tag{2.5}$$

As we said before this,the interaction term and so this model at $d = 3$, becoming strongly coupled as we go to IR regime. So we are not allowed to use ordinary perturbation methods. Here we will use ϵ-expansion and $\frac{1}{N}$ expansion to find new window for study the system.

The idea behind ϵ-expansion is really simple. Naively this method say although we are banned to use perturbation theory in $d < 4$, but let's see what happens if we cross the line just a little bit at $d = 4 - \epsilon$? after calculation by the mean of perturbation, you can use $\epsilon = 1$ to achieve the results for $d = 3$. Experimental and numerical results approve that we are not wrong and we can do this machinery.

Another approach is $\frac{1}{N}$ expansion and taking large N limit that uses some auxiliary field (non-dynamical) and surprisingly these two approach match really nice. Although both methods will not converge, but they provide asymptotic series. Researchers, are trying to find new ways to study non-perturbative theories. One of the other solutions, is to use conformal bootstrap (for a short review you can check Mr. Ameri's term-paper or for the case of $O(N)$ vector model see [9]). This method has been studied models with $2 < d < 6$.

2.2 ϵ-expansion approach

Consider N number of real scalar fields, we show with ϕ^i:

$$\phi^i = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} \quad i = 1, \ldots, N \tag{2.6}$$

This field have $O(N)$ invariant action [1]:

$$S = \int d^3x \left(\frac{1}{2}(\partial\phi^i)^2 + \frac{\lambda}{4}(\phi^i\phi^i)^2\right) \tag{2.7}$$

Once again, using dimensional analysis, we see for $d < 4$ the interaction term is relevant and so in IR limit $\Lambda \to IR$, we get strongly coupled interaction. Using Wilson-Fisher ϵ-expansion or in short ϵ-expansion, we set $d = 4 - \epsilon$ and compute beta function as ordinary ϕ^4 theory (one-loop):

$$\beta_\lambda = -\epsilon\lambda + (N+8)\frac{\lambda^2}{8\pi^2} \tag{2.8}$$

[1] the $O(N)$ group has elements like $g_i \in O(N)$ where $g_i^T g_i = 1$. Each ϕ^i goes to $g_i\phi^j$, so for terms like $\phi^j\phi^j$ we have: $\phi^j\phi^j \to (g_i\phi^j)^T(g_i\phi^j) = \phi^{jT}\phi^j = \phi^j\phi^j$.

Note that factor that contains N is come from symmetry factors of Feynman diagrams during calculation (see [8] for review). It is easy to see the IR fixed point is:

$$\lambda_* = \frac{8\pi^2}{N+8}\epsilon \tag{2.9}$$

Actually we could find this beta function, using the results from the end of previous chapter, by imposing some considerations.

We can go further and calculate anomalous dimension:

$$\gamma_\phi = \frac{N+2}{4(N+8)^2}\epsilon^2 + O(\epsilon^3) \tag{2.10}$$

$$\gamma_{\phi^2} = \frac{N+2}{N+8}\epsilon + O(\epsilon^2) \tag{2.11}$$

So by the definition of scaling dimension, we have for ϕ^i and $\phi^i\phi^i$ [2] [3]:

$$\Delta_\phi = \frac{d-2}{2} + \gamma_\phi = 1 - \frac{\epsilon}{2} + \frac{N+2}{4(N+8)^2}\epsilon^2 + O(\epsilon^3) \tag{2.12}$$

$$\Delta_{\phi^2} = d - 2 + \gamma_{\phi^2} = 2 - \frac{6}{N+8}\epsilon + O(\epsilon^3) \tag{2.13}$$

If we go to $d > 4$, our interaction coupling is irrelevant, we get free theory fixed-point. If we take $d = 4 + \epsilon$ and repeat the calculations, we can have:

$$\lambda_* = -\frac{8\pi^2}{N+8}\epsilon\,. \tag{2.14}$$

Once again we can compute anomalous dimension or we can simply set $\epsilon \to -\epsilon$ and find the same results.

2.3 $\frac{1}{N}$ expansion and large N limit

Some theories in special scales of energy, have weak coupling (UV regimes or high-energy or equivalently small distances) but the story in IR regime is different and they become strongly coupled so we are not allowed to use perturbation. For example the Yang-Mils theory in $d = 4$ have strongly couple behavior in low energy. To face these theories we need to invent new methods which called non-perturbative methods. One of these methods is large N expansion. Here N can be rank of symmetry group or rank of representation.

One of the simplest case to start with is our case i.e. $O(N)$ vector model. For a while consider the massive real scalar field with the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i(x))^2 + \frac{m_0^2}{2}\phi_i^2 + \frac{\lambda}{4!}(\phi_i^2)^2 \tag{2.15}$$

We then use our trick and take the coupling to $\lambda \to \frac{g}{N}$. We know that same indices are summed so for a moment we drop the indices:

$$\mathcal{L} \to \frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{m_0^2}{2}\phi^2 + \frac{g}{N4!}(\phi^2)^2 \tag{2.16}$$

First we mention some point from diagrammatic, illustration of this method, and next we see what is going on in path integral approach. The usual vertex of ϕ^4 theory becomes like:

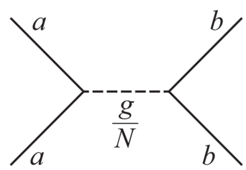

Figure 2.2: The ϕ^4 interaction when we change coupling as $\frac{g}{N}$. For clarity of what is happening there we spited the vertex. This image has been taken from [4]

Actually one can split the snail diagram of one-loop self energy in two ways:

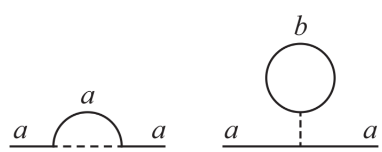

Figure 2.3: Snail diagrams or one-loop corrections to propagator. This image has been taken from [4]

These two last diagrams have two different factor of coupling. The left hand side diagram has a factor of $\frac{g}{N}$ and when we take the large N limit $N \to \infty$, this diagram fades away. The right hand side one, have a factor $\frac{g}{N}$ too, but an extra factor of N, coming from sum over indices in the loop part. So when we go to large N, the second diagram has a factor $\frac{g}{N}N = g$ and will not die in large N. We go further and see what is happening at higher loop corrections. For instance, at the two-loop level, we obtain these diagrams:

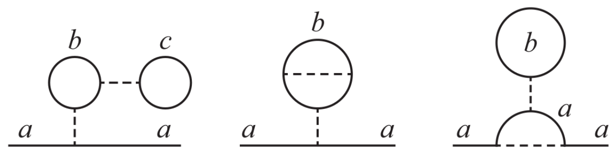

Figure 2.4: Two loop corrections to propagator. This image has been taken from [4]

Just as previous case, the only surviving diagram is the diagram on the left. We can see a pattern here, in each order of correction, the surviving diagram is the one whose the number of loops is equal to the number of order in corrections. One may ask that, we see diagrams with loops will not die, but we again have g so what is the point of this process? here we should mention that when we change the coupling we impose to respect 't Hooft coupling condition which makes our diagram more smooth as before [1] [18]:

$$g \equiv \lambda N \rightarrow N \rightarrow \infty, g \rightarrow \text{fixed value} \tag{2.17}$$

In path integral approach, we will introduce an auxiliary (non-dynamical) field σ which called Hubbard–Stratonovich field. Then we integrate out our fundamental field ϕ, and get an effective action.

$$\mathcal{Z} = \int \mathcal{D}\phi exp(-\int d^d x(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{m_0^2}{2}\phi^2 + \frac{g}{2N}(\phi^2)^2)) \tag{2.18}$$

$$= \int \mathcal{D}\sigma \int \mathcal{D}\phi exp(-\int d^d x(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{m_0^2}{2}\phi^2 - \frac{N}{2g}\sigma^2 + \sigma\phi^2)) \tag{2.19}$$

$$= \int \mathcal{D}\sigma (\det(-\partial^2 + m_0^2 + 2\sigma))^{\frac{-N}{2}} exp(+\int d^d x \frac{N}{2g}\sigma^2) \tag{2.20}$$

Using the identity for Gaussian integrals[2] we have:

$$\mathcal{Z} = \int \mathcal{D}\sigma exp(-N S_{eff}(\sigma)) \tag{2.21}$$

$$S_{eff}(\sigma) = \frac{1}{2} Tr \ln(-\partial^2 + m_0^2 + 2\sigma) - \frac{1}{2g}\int d^d x \sigma^2 \tag{2.22}$$

[2] $\int d^n x e^{-xAx} = \frac{\pi^{\frac{n}{2}}}{\det A} = \pi^{\frac{n}{2}} exp(-\frac{1}{2}\ln\det A)$

For a more detailed reference on the subject see [19] and [13]. Next, we will explore to see what happens to our initial massless ϕ^4 theory. From above with some changing in factors we can have:

$$S = \int d^d x \left(\frac{1}{2}(\partial\phi^i)^2 + \frac{1}{2}\sigma\phi^i\phi^i - \frac{\sigma^2}{4\lambda} \right) \tag{2.23}$$

If compute the equation of motion for auxiliary field we see $\sigma = \lambda\phi^i\phi^i$, plug this in our action, we get our initial action for massless scalar field. Again by integrating out fundamental field ϕ^i, we can compute effective action as:

$$\begin{aligned} Z &= \int D\phi D\sigma\, e^{-\int d^d x \left(\frac{1}{2}(\partial\phi^i)^2 + \frac{1}{2\sqrt{N}}\sigma\phi^i\phi^i\right)} \\ &= \int D\sigma\, e^{\frac{1}{8N}\int d^d x d^d y\, \sigma(x)\sigma(y)\, \langle\phi^i\phi^i(x)\phi^j\phi^j(y)\rangle_0 + \mathcal{O}(\sigma^3)} \times e^{\int d^d x \frac{\sigma^2}{4\lambda} + O(\sigma^3)} \end{aligned} \tag{2.24}$$

Here we used large N limit and computed trace of logarithm. The subscript 0 indicates the expectation value in the free theory.

For the propagator of fundamental field:

$$G(x-y) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip(x-y)}}{p^2} \tag{2.25}$$

So for the quartic terms by considering two choice between $\phi^i(x)$ and $\phi^j(y)$, and N choice for each of indices we have:

$$\langle \phi^i(x)\phi^i(x)\phi^j(y)\phi^j(y)\rangle_0 = 2N[G(x-y)]^2 \tag{2.26}$$

Next, we apply the Fourier transform and switch to the momentum domain:

$$|G(x-y)|^2 = \int \frac{d^d p}{(2\pi)^d} e^{ip(x-y)} \tilde{G}(p) \tag{2.27}$$

$$\tilde{G}(p) = \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2(p-q)^2} = -\frac{(p^2)^{d/2-2}}{2^d (4\pi)^{\frac{d-3}{2}} \Gamma(\frac{d-1}{2}) \sin(\frac{\pi d}{2})} \equiv \frac{-2}{\tilde{c}_\sigma}(p^2)^{\frac{d}{2}-2} \tag{2.28}$$

Next, we use the Fourier transform and the effective action we derived earlier, we can see the effective quadratic action for σ as:

$$S_2 = -\int d^d P (\frac{1}{2}\sigma(p)\sigma(-p)(\frac{N}{\tilde{c}_\sigma}(p^2)^{\frac{d}{2}-2} + \frac{1}{2\lambda})) \tag{2.29}$$

As you see, if we want to compare kinetic term and σ^2 term in $d < 4$, when we go to IR regime, the kinetic term will grow more and more so we can drop σ^2 term. In short we can work with this action from now on:

$$S = \int d^d x \left(\frac{1}{2}(\partial \phi^i)^2 + \frac{1}{2\sqrt{N}} \sigma \phi^i \phi^i \right) \tag{2.30}$$

Note that here we have changed field σ by a factor of $\frac{1}{\sqrt{N}}$. From action S_2 one can find σ field two point function is:

$$\langle \sigma(p)\sigma(-p)\rangle = \tilde{c}_\sigma (p^2)^{2-\frac{d}{2}} = 2^{d+1}(4\pi)^{\frac{d-3}{2}} \Gamma(\frac{d-1}{2}) \sin(\frac{\pi d}{2})(p^2)^{2-\frac{d}{2}} \tag{2.31}$$

Easily you can use Fourier transform again and go to space time:

$$\langle \sigma(x)\sigma(y)\rangle = \frac{2^{d+2}\Gamma\left(\frac{d-1}{2}\right)\sin(\frac{\pi d}{2})}{\pi^{\frac{3}{2}}\Gamma\left(\frac{d}{2}-2\right)} \frac{1}{|x-y|^4} \equiv \frac{C_\sigma}{|x-y|^4} \tag{2.32}$$

It is time that we can use large N perturbation theory, using the propagators for field σ and fundamental fields ϕ^i.

Similar to the situation of the ϵ-expansion, we want to compute anomalous dimensions and then scaling dimensions. To do so, we have one loop correction to fundamental field propagator as:

$$\frac{1}{N} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(p-q)^2} \frac{\tilde{C}_\sigma}{(q^2)^{\frac{d}{2}-2+\delta}} \tag{2.33}$$

In this equation δ is regulator. If we continue and calculating this integral, one can follow to get (see [3] and it's appendix A for integration):

$$\Delta_\phi = \frac{d}{2} - 1 + \frac{1}{N}\eta_1 + \frac{1}{N^2}\eta_2 + \dots \tag{2.34}$$

For η_1 and η_2 we have (see [3],[17] and [16]):

$$\eta_1 = \frac{\tilde{C}_\sigma (d-4)}{(4\pi)^{\frac{d}{2}} d\Gamma(\frac{d}{2})} = \frac{2^{d-3}(d-4)\Gamma\left(\frac{d-1}{2}\right)\sin\left(\frac{\pi d}{2}\right)}{\pi^{\frac{3}{2}}\Gamma\left(\frac{d}{2}+1\right)} \tag{2.35}$$

$$\eta_2 = 2\eta_1^2 \left(f_1 + f_2 + f_3\right) \tag{2.36}$$

Here we define:

$$f_1 = v'(\mu) + \frac{\mu^2+\mu-1}{2\mu(\mu-1)}\,, \quad f_2 = \frac{\mu}{2-\mu}v'(\mu) + \frac{\mu(3-\mu)}{2(2-\mu)^2} \tag{2.37}$$

$$f_3 = \frac{\mu(2\mu-3)}{2-\mu}v'(\mu) + \frac{2\mu(\mu-1)}{2-\mu} \tag{2.38}$$

$$v'(\mu) = \psi(2-\mu) + \psi(2\mu-2) - \psi(\mu-2) - \psi(2)\,, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}\,, \quad \mu = \frac{d}{2} \tag{2.39}$$

Once again if we set $d = 4-\epsilon$, we can get exactly the same result we had for ϵ-expansion approach. More over one can compute [17]:

$$\Delta_\sigma = 2 + \frac{1}{N}\frac{4(d-1)(d-2)}{d-4}\eta_1 + \mathcal{O}(\frac{1}{N^2}) \tag{2.40}$$

We can check that for this scaling dimension, we again have the same as before by setting $d = 4-\epsilon$.

Next, we will go further and examine the case when we set $d = 6-\epsilon$. There is no specific prohibition as we respect the unitarity bound (for spin-zero operators we have $\Delta \geq \frac{d-2}{2}$). In this case we see:

$$\Delta_\phi = 2 - \frac{\epsilon}{2} + \left(\frac{1}{N} + \frac{44}{N^2} + \frac{1936}{N^3} + \dots\right)\epsilon - \left(\frac{11}{12N} + \frac{835}{6N^2} + \frac{16352}{N^3} + \dots\right)\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{2.41}$$

Also we have:

$$\Delta_\sigma = 2 + \left(\frac{40}{N} + \frac{6800}{N^2} + \dots\right)\epsilon - \left(\frac{104}{3N} + \frac{34190}{3N^2} + \dots\right)\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{2.42}$$

We recall these results in the next section and we will see that they match exactly to the results of cubic theory in $d = 6-\epsilon$ at it's IR fixed point.

At the end of this section let me compare the results from $\frac{1}{N}$ expansion and from bootstrap method for the case of $d = 3$. Using $\frac{1}{N}$ expansion method people have found:

$$\Delta_\phi = \frac{1}{2} + \frac{4}{3\pi^2}\frac{1}{N} - \frac{256}{27\pi^4}\frac{1}{N^2} + O(\frac{1}{N^3})$$
$$\Delta_s = 2 - \frac{32}{3\pi^2}\frac{1}{N} + \frac{32(16-27\pi^2)}{27\pi^4}\frac{1}{N^2} + O(\frac{1}{N^3})$$

Here s is the same $\phi^i\phi^i$.On the other hand from bootstrap methods we get:

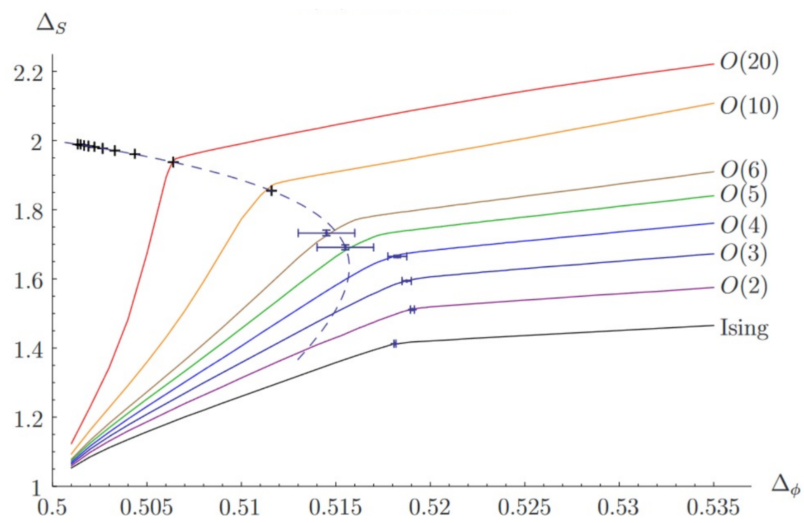

Figure 2.5: The kink represents the 3d critical Ising model for N=1. Here, we show N = 1, 2, 3, 4, 5, 6, 10, 20. The blue error bars show the most accurate estimates of the operator dimensions using analytical and numerical methods. The black crosses show the predictions from $\frac{1}{N}$ expansion. The dashed line interpolates the large-N prediction. This image has been taken from [9]

.

Which perfectly matches the results from $\frac{1}{N}$ expansion method for the position of kinks.

2.4 Cubic theory

In previous section we studied the ϕ^4 theory in $2 < d < 4$. now we are interested in seeking the evidence for $d > 4$. First thing we need to consider is in $d > 4$ term $(\phi^i\phi^i)^2$ becomes irrelevant, so in these areas we have free theory in IR at Gaussian critical-point. Any way, we may ask how much we can go? the answer is because of scaling dimension $\Delta_{\phi^2} = 2 + O(\frac{1}{N})$ (which we have computed for any d) we can not pass $d = 6$, to respect unitarity bound $\frac{d-2}{2}$. So our playground will be $2 < d < 6$.

Now we will look at a Lagrangian with cubic terms:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi^i)^2 + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2}{6}\sigma^3 \tag{2.43}$$

Which is $O(N)$ symmetric theory. Interacting terms in $d < 6$ are relevant, so the theory will flow from Gaussian critical-point to an IR interacting critical-point when we lower the scale of energy. The Feynman rules in this theory for propagator and vertex of phi^i fields and σ field are shown below:

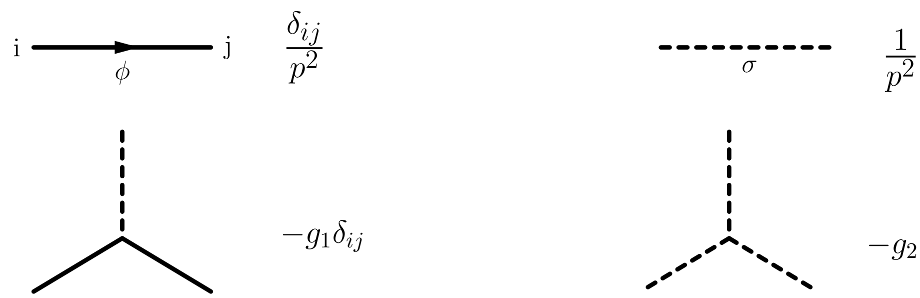

Figure 2.6: Propagators and vertexes of fields. Image has been taken from [3]

Using these vertexes and compute one-loop corrections to β functions (for large N limit), leads to following results:

$$\beta_1 = -\frac{\epsilon}{2}g_1 + \frac{Ng_1^3}{12(4\pi)^3} \tag{2.44}$$

$$\beta_2 = -\frac{\epsilon}{2}g_2 + \frac{-4Ng_1^3 + Ng_1^2 g_2}{4(4\pi)^3} \tag{2.45}$$

And we can find critical-point couplings (in large N):

$$g_1^* = \sqrt{\frac{6\epsilon(4\pi)^3}{N}} \tag{2.46}$$

$$g_2^* = 6g_1^* = 6\sqrt{\frac{6\epsilon(4\pi)^3}{N}} \tag{2.47}$$

Next, we will compare the results from the preceding section about $d = 6 - \epsilon$ and the new results but in the same d from cubic theory. We have anomalous dimension and scaling dimension for ϕ^i as:

$$\gamma_\phi = \frac{1}{(4\pi)^3}\frac{g_1^2}{6} \tag{2.48}$$

$$\Delta_\phi = \frac{d-2}{2} + \gamma_\phi = 2 - \frac{\epsilon}{2} + \frac{\epsilon}{N} + \frac{44\epsilon}{N^2} + \frac{1936\epsilon}{N^3} + \dots \tag{2.49}$$

Note that here we have used exact beta function in powers of $\frac{1}{N}$ to get O($\frac{1}{N}$) corrections for couplings. And for σ field we have:

$$\gamma_\sigma = \frac{Ng_1^2 + g_2^2}{(4\pi)^3 12} \tag{2.50}$$

$$\Delta_\sigma = 2 + \frac{40\epsilon}{N} + \frac{6800\epsilon}{N^2} + \dots \tag{2.51}$$

These results clearly agree with those from the ϕ^4 theory. We call this cubic theory in $d = 6-\epsilon$ as the UV completion of the quartic theory in $d > 4$. For detailed calculation see [3]. In the following figure we show the entire accessible range $2 < d < 6$:

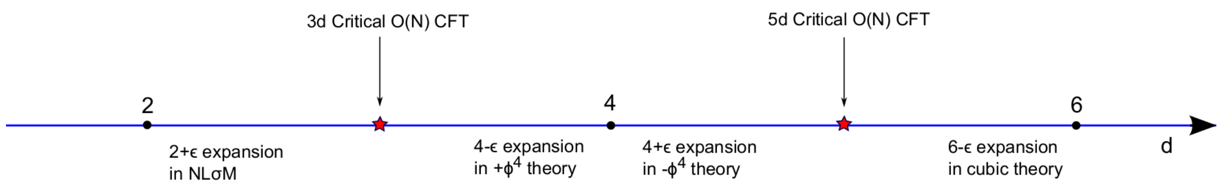

Figure 2.7: Different theories in different dimension between 2 to 6. Near $d = 2 + \epsilon$ we have non-linear sigma model which was not in our discussion but one can see [10] and [11] for more details.

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