String Field Theory I

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Contents

Motivations

Why SFT?

- It is a field theory (second quantization)
- More rigorous and constructive formulation
- proof of dualities
- proof of AdS/CFT correspondence
- dynamics of compactifications
- higher structure of string theory
- ... [1]

[1] H. Erbin, String Field Theory: A Modern Introduction, vol. 980 of Lecture Notes in Physics. 3, 2021, 10.1007/978-3-030-65321-7

References

The references I mainly used for these slides are:

- A Review on tachyon condensation in open string field theories-Kazuki Ohmori (main reference)
- String Field Theory – A Modern Introduction-Harold Erbin
- Four Lectures on Analytic Solutions in Open String Field Theory-Theodore Erler
- Level Truncation Approach to Open String Field Theory-Matěj Kudrna
- Non-commutative geometry and string field theory-Edward Witten
- Analytical Solutions of Open String Field Theory -Ehud Fuchs, Michael Kroyter
- Universality of the Tachyon Potential-Ashoke Sen
- An Overview of Tachyon Condensation and SFT-L. Bonora

What is a string field?

defenition: string field theory is the second quantization of string theory. by this definition, functionals $\Phi\left[X^{\mu}, b, c\right] = \langle X^{\mu}, b, c| \, |\Phi\rangle$ are those string fields. in the first quantized theory, base space is world sheet parameters and target space is space-time coordinates. here in second quantized theory base space is space-time (and ghosts).
Note: there is different between this second quantize theory and ordinary QFT, that is in QFT we deal with point particles but here we have a field theory for extended objects.
Note: we consider Φ for open string fields, and Ψ for closed String fields.

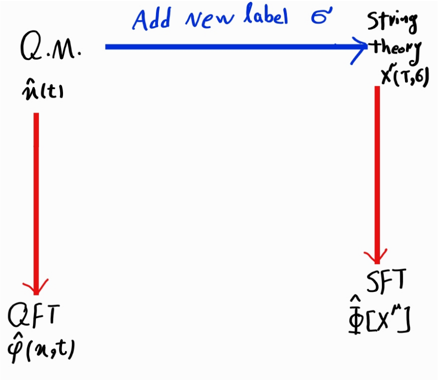

Figure: different theories by considering different definition of labels and operators.

let us recall the Hilbert space $\mathcal{H}$ of first quantized string theory. any state is constructed by the action of negative modes α^{mu}_{-n}, b_{-m} and c_{-l} on vacuum $|\Omega\rangle$. the relation between this vacuum and SL(2,R) invariant vacuum $|0\rangle$ is:

$$|\Omega\rangle = c_1 |0\rangle$$

also we have these properties:

$$\begin{aligned}
&\alpha^{\mu}_{n} |\Omega\rangle = 0 \qquad n > 0 \\
&b_n |\Omega\rangle = 0 \qquad n \geq 0 \\
&c_n |\Omega\rangle = 0 \qquad n > 0 \\
&p^{\mu} |\Omega\rangle \propto \alpha^{\mu}_{0} |\Omega\rangle
\end{aligned}$$

A basis for Hilbert space $\mathcal{H}$ is provided by the collection of states of the form:

$$\alpha^{\mu_1}_{-n_1}\dots\alpha^{\mu_i}_{-n_i}b_{-m_1}\dots b_{-m_j}c_{-l_1}\dots c_{-l_k}\left|\Omega\right\rangle$$

then any arbitrary state $\left|\Phi\right\rangle \in \mathcal{H}$ can be expanded as:

$$\left|\Phi\right\rangle = (\phi(x) + A_\mu(x)\alpha^\mu_{-1} + B_{\mu\nu}(x)\alpha^\mu_{-1}\alpha^\nu_{-1} + \dots)c_1\left|0\right\rangle \equiv \Phi(z=0)\left|0\right\rangle$$

where $\left|\Phi\right\rangle$ is string field state and vertex operator Φ is string field.

Physical conditions and free action of SFT

Let's check the physical conditions for string field state in BRST quantization approach:

- $Q_B \left|\Phi\right\rangle = 0$
- $\#_{ghost} \left|\Phi\right\rangle = 1$

then by choosing closed states,our physical Hilbert space becomes:

$$\mathcal{H}_{physical} = \frac{\mathcal{H}^1_{closed}}{\mathcal{H}^1_{exact}}$$

this is exactly the same as put away null (spurious) states from $\mathcal{H}_{physical}$ in old covariant quantization (OCQ) approach. if we choose free action S_0 of SFT as follow:

$$S_0 = \left\langle\Phi\right| Q_B \left|\Phi\right\rangle$$

It's e.o.m. gives the physical condition $Q_B \left|\Phi\right\rangle = 0$.

Witten's cubic string field theory

Witten proposed an axiomatic theory to formulate field theory of open strings with these axioms:

- multiplication:

  define $*$ product between String fields A and B as $A * B$;

- Associativity:

  $A * (B * C) = (A * B) * C$;

- Existence of an odd derivation:
  $Q_B(A * B) = Q_B A * B + (-1)^{\#A} A * Q_B B$

- Nilpotency:

  $Q_B^2 = 0$

- Integration:

  Integration maps string field A to a complex number $\int A \in \mathbb{C}$and it is linear, $\int (A + B) = \int A + \int B$. more over this integration satisfies $\int A * B = (-1)^{\#A\#B} \int B * A$. Also $\int Q_B A = 0$

[2]

[2] E. Witten, "Noncommutative Geometry and String Field Theory," Nucl. Phys. B 268, 253– 294 (1986)

Before exploring the full action, it is good to see a comparison between differential forms and Witten's string fields:

	Algebra of Witten's SFT	Differential forms
elements	string field	differential p-form
degree	$(-1)^{\#A}$	$(-1)^p$
multiplication	$*$-product	$\wedge$-product
derivation	Q_B	exterior derivative d
integration	$\int$	$\int$ on a p-dimensional manifold

Table: Dictionary between differential forms and String fields.

Multiplication and Integration

Multiplication: We should interpret $*$-product as gluing two half-strings together. take two string S and T which their excitations are described by string fields A and B. we label each one of strings by parameter σ from 0 to π with midpoint in $\frac{\pi}{2}$. then we glue right part of string S to the left part of string T. the result is a string like U as you see:

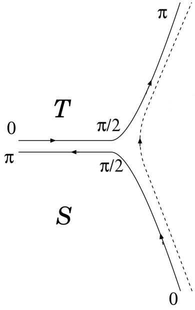

Figure: $A * B$

Integration: by using the definition of previous slide, we interpret the integration of string fields $\int A * B$ as the procedure of gluing remaining sides of string S and T:

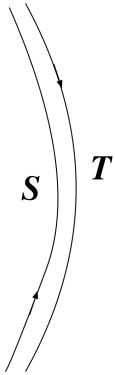

Figure: $\int A * B$

Witten vertex

Using the previous definition of $*$ and $\int$, we define n-string interaction vertex as $\int \Phi_1 * \ldots \Phi_n$, where Φ_i denotes a string field in i-th string Hilbert space. such an interaction is called Witten vertex.

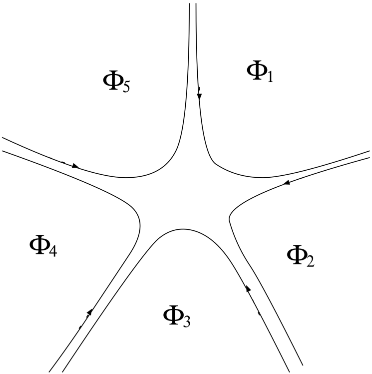

Figure: Witten vertex (n=5).

Full action

Using the definitions:

$$S_0 = \langle\Phi| Q_B |\Phi\rangle = \int \Phi * Q_B \Phi$$

and for interaction part:

$$S_{int} = S_n = \int \underbrace{\Phi * \cdots * \Phi}_{\text{n times}}$$

Note: Here we have $\#\Phi = 1$, $\#Q_B = 1$ and $\#* = 0$ as we see next for calculation we go to complex unit disk, and based on Riemann-Roch theorem, the correlation function vanish unless the equation

$$(N_c) - (N_b) = 3\chi$$

holds for riemann surface with Euler characteristics χ (N_b and N_c are number of ghosts.). so only S_3 can be nonzero in the case of disk ($\chi = 1$).

Gauge invariance

the full action has the form:

$$S = -\frac{1}{g_0^2}(\frac{1}{2\alpha'}\int \Phi * Q_B \Phi + \frac{1}{3}\int \Phi * \Phi * \Phi)$$

where g_0 is a dimensionfull parameter. this action is invariant under following infinitesimal gauge transformation:

$$\delta\Phi = Q_B\Lambda + g_0(\Phi * \Lambda - \Lambda * \Phi)$$

where Λ is gauge parameter with $\#\Lambda = 0$.

Chern-Simons action

Witten action has the same structure as a Chern-Simons action:

$$S(A) = \frac{1}{2}\int_M A \wedge dA + \frac{1}{3}\int_M A \wedge A \wedge A$$

	Witten's open SFT	Chern-Simons
elements	string field	gauge field A
degree	$(-1)^{\#\Phi}$	$(-1)^p$
multiplication	$*$-product	$\wedge$-product
derivation	Q_B	exterior derivative d
integration	$\int$	$\int$ on a p-dimensional manifold
gauge parameter	ϵ	Λ with $\# = 0$

Table: Dictionary between Witten's open SFT and Chern-Simons theory.

Usual mapping in string theory, from world-sheet parameters to complex plane:

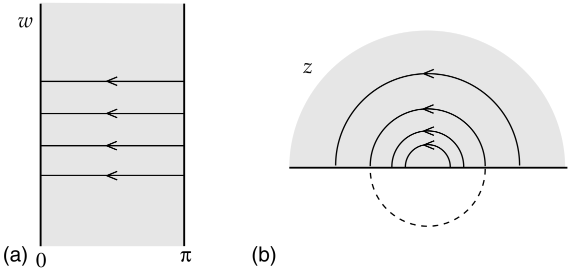

Fig. 2.5. Open string coordinates. (a) Equal time contours in the w-plane. (b) The same contours in the z-plane. The dashed line shows the extension of one contour, as used in the doubling trick.

Figure: This image has been taken from "String Theory V1-Book by J. Polchinski " .

$$\mathbf{z} = \mathbf{e}^{(\tau - \mathbf{i}\sigma)}, \qquad \mathbf{0} \leq \sigma \leq \pi, \qquad -\infty < \tau < \infty$$

Rewrite the action

Our last version of SFT action is not suitable for calculations, so we need to go to complex plane and use CFT techniques. for cubic part, the main idea is to map three upper half-disks, to one disk.

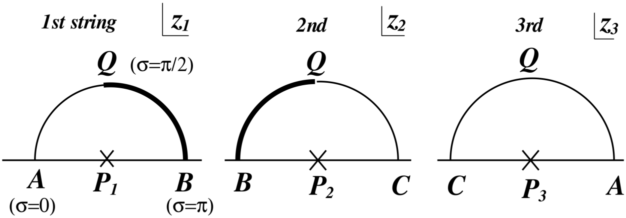

Figure: three strings joining the interaction.

each one of world sheets, has been created at $t = -\infty$ which is corresponds to $z_i = 0$ by the action of vertex operator P_i, and now at $t = 0$ they interact ($|z_i| = 1$).

Now we want to map these three world sheet (with local coordinates z_i's) into one disk (with coordinate $\boldsymbol{\zeta}$). we should do transformations with following properties:

- It maps the interaction point to the center of unit disk ($\boldsymbol{\zeta} = 0$)
- open string boundaries, are mapped to the boundary of unit disk.

for simplicity consider second open string. the transformation

$$z_2 \to w = h(z_2) = \frac{1 + iz_2}{1 - iz_2}$$

satisfies above condition. but since the angel $\angle BQC = 180^\circ$, three half disk can't be put side by side to form a disk. so we need an extra transformation:

$$w \to \boldsymbol{\zeta} = \eta(w) = w^{\frac{2}{3}}$$

which maps with right angel 120°.

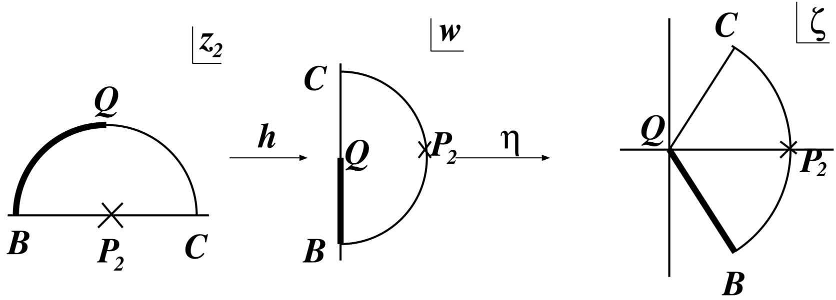

Figure: from upper half disk to one third of a unit disk.

All we have to do for other two world sheet is to rotate them by angels $\pm 120^\circ$. these steps will be achieved by:

$$g_1(z_1) = e^{-\frac{2\pi i}{3}}(\frac{1+iz_1}{1-iz_1})^{\frac{2}{3}}$$

$$\eta o h(z_2) = g_2(z_2) = (\frac{1+iz_2}{1-iz_2})^{\frac{2}{3}}$$

$$g_3(z_3) = e^{\frac{2\pi i}{3}}(\frac{1+iz_3}{1-iz_3})^{\frac{2}{3}}$$

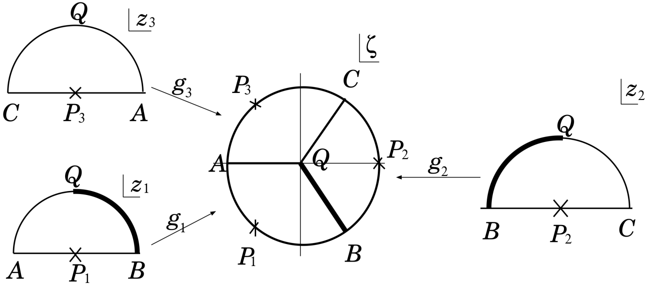

Figure: 3-string vertex.

using these mappings, we have:

$$\int \Phi * \Phi * \Phi = \langle g_1 o\Phi(0) g_2 o\Phi(0) g_3 o\Phi(0)\rangle$$

where $\langle \ldots \rangle$ is correlator on unit disk constructed above. $g_i o\Phi(0)$ is conformal transformation of $\Phi(0)$ by g_i. if Φ is a primary field of weight h this transforms as:

$$g_i o\Phi(0) = (g_i'(0))^h \Phi(g_i(0))$$

now let's construct a transformation which maps unit disk to upper half plane:

$$z = h^{-1}(\zeta) = -i\frac{\zeta - 1}{\zeta + 1}$$

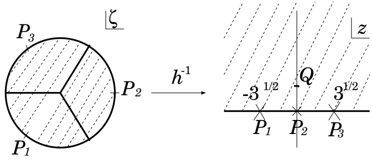

Figure: from unit disk to UHP.

For cubic term we have:

$$\int \Phi * \Phi * \Phi = \langle f_1 o\Phi(0) f_2 o\Phi(0) f_3 o\Phi(0)\rangle$$

$$f_i(z_i) = h^{-1} o g_i(z_i)$$

and finally for quadratic term we should use these two transformations:

$$f_1(z_1) = h^{-1}\left(\frac{1+iz_1}{1-iz_1}\right) = z_1 = id(z_1)$$

$$f_2(z_2) = h^{-1}\left(\frac{1+iz_2}{1-iz_2}\right) = -\frac{1}{z_2} \equiv \mathcal{I}(z_2)$$

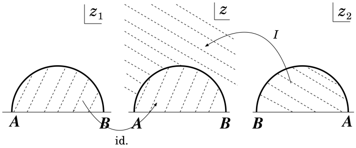

Figure: two point vertex.

so the quadratic term is:

$$\int \Phi * Q_B \Phi = \langle \mathcal{I} o\Phi(0) Q_B \Phi(0)\rangle$$

Gauge fixing

In the previous slides we get action in the form:

$$S = -\frac{1}{g_0^2}\left(\frac{1}{2\alpha'}\left\langle \mathcal{I} o \Phi(0) Q_B \Phi(0)\right\rangle + \frac{1}{3}\left\langle f_1 o \Phi(0) f_2 o \Phi(0) f_3 o \Phi(0)\right\rangle\right)$$

now we want to do gauge-fixing by using a special choice so-called Feynman-Siegel gauge:

$$b_0 \left|\Phi\right\rangle = 0$$

this gauge is a good choice, because:

- there are no residual gauge transformation.
- we can always choose this gauge.

Level truncation

First we expand $\# = 1$ string field state:

$$|\Phi\rangle = \int \mathrm{d}^d k(\phi + A_\mu \alpha^\mu_{-1} + i\alpha b_{-1}c_0 + \frac{i}{\sqrt{2}} B_\mu \alpha^\mu_{-2} + \frac{1}{\sqrt{2}} B_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + \beta_0 b_{-2}c_0 + \beta_1 b_{-1}c_{-1} + i\kappa_\mu \alpha^\mu \alpha^\mu_{-1} b_{-1}c_0 + \dots)c_1 |0\rangle$$

we know that:

$$L_0^{tot} = \alpha' p^2 + \sum_{n=1}^{\infty} \alpha^\mu_{-n}\alpha_{\mu n} + \sum_{n=-\infty}^{\infty} n c_{-n} b_n - 1$$

we call sum of second and third term of L_0^{tot}, level number. by saying level(M,N) truncation, we mean that we take level $\leq$ M terms for string field state and take level $\leq$ N terms for string field action.

Example

We choose level(1,3) truncation and calculate terms of the action. our string field state is:

$$|\Phi\rangle = \int \mathrm{d}^d k(\phi(k) + A_\mu(k)\alpha^\mu_{-1})c_1 |0\rangle$$
$$= \int \mathrm{d}^d k(\phi(k)c(0) + \frac{i}{\sqrt{2\alpha'}}A_\mu(k)c\partial X^\mu(0)) |k\rangle$$

where in the second line we expressed string field state in term of vertex operators. here vertex operator of $|k\rangle$ is $|k\rangle = e^{ikX(0)}|0\rangle$. conformal transformation of vertex operators are given by:

$$\mathcal{I}o(ce^{ikX}(\epsilon)) = (\frac{1}{\epsilon^2})^{-1+\alpha' k^2} ce^{ikX}(-\frac{1}{\epsilon})$$
$$\mathcal{I}o(c\partial X^\mu e^{ikX}(\epsilon)) = (\frac{1}{\epsilon^2})^{\alpha' k^2} c\partial X^\mu e^{ikX}(-\frac{1}{\epsilon})$$

After a lot of calculations, the quadric term of action becomes:

$$S_{quad} = (2\pi)^d \int \mathrm{d}^d k((k^2 - \frac{1}{\alpha'})\phi(-k)\phi(k) + k^2 A_\mu(-k)A^\mu(k))$$

[3]

[3] K. Ohmori, "A Review on tachyon condensation in open string field theories,"

Fourier-transforming to the position space as:

$$\phi(k) = \int \frac{\mathrm{d}^d x}{(2\pi)^d}\phi(x)e^{-ikx}, \qquad A_\mu(k) = \int \frac{\mathrm{d}^d x}{(2\pi)^d}A_\mu(x)e^{-ikx}$$

so we have:

$$S_{quad} = \frac{1}{g_0^2}\int \mathrm{d}^d x(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2\alpha'}\phi^2 - \frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu)$$

proceeding this method, after tons of calculations for cubic part and considering terms with $\phi\phi\phi$ and ϕAA form because of twist symmetry, we have full action in the form of:

$$S = S_{quadric} + S_{\phi\phi\phi} + S_{\phi AA} = \frac{1}{g_0^2}\int \mathrm{d}^d x(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2\alpha'}\phi^2 - \frac{1}{3}(\frac{3\sqrt{3}}{4})^3\tilde{\phi}^3$$
$$- \frac{-1}{2}\partial_\mu A_\nu\partial^\mu A^\nu - \frac{3\sqrt{3}}{4}\tilde{\phi}\tilde{A}_\mu\tilde{A}^\mu - \frac{3\sqrt{3}\alpha'}{8}(\partial_\mu\partial_\nu\tilde{\phi}\tilde{A}^\mu\tilde{A}^\nu + \tilde{\phi}\partial_\mu\partial_\nu\tilde{A}^\mu\tilde{A}^\nu$$
$$- 2\partial_\mu\tilde{\phi}\partial_\nu\tilde{A}^\mu\tilde{A}^\nu)$$

where we have: $\tilde{\phi}(x) = exp(-\alpha' \ln \frac{4}{3\sqrt{3}}\partial_\mu\partial^\mu)\phi(x)$

Universality of the tachyon potential

one can get the equation of motion of SFT by variation of action respect to $|\Phi\rangle$ vanishes ($Q_B\Phi + \Phi * \Phi = 0$). also we can have e.o.m. in component form by decomposing $|\Phi\rangle$ and setting to the zero the variation of action with respect to each coefficient of expansion.
if some component fields like ψ, always enter the action quadratically or in higher order (S contains no linear terms in ψ) we always have at least one factor of ψ in e.o.m., which is satisfied by setting $\psi = 0$. we can set all such fields to zero in our calculations.

now we want to know such fields in the language of BCFT:
a string field is an element of Hilbert space $\mathcal{H}^1$ of #1. we decompose this space to $\mathcal{H}^1 = \mathcal{H}^1_1 \oplus \mathcal{H}^1_2$, which $\mathcal{H}^1_1$ contains all states made of acting c_n, b_n and L_n^{matter} on vacuum $|0\rangle$. this subspace contains field with zero momentum such as tachyon state $c_1 |0\rangle$. on the other hand in $\mathcal{H}^1_2$ we have all other states in $\mathcal{H}^1_1$. they are state with non-zero momentum and states made of the action of c_n, b_n and L_n^{matter} on primary states of $h > 0$. (we can easily prove that string fields in $\mathcal{H}^1_2$ are always non-linear in action.)
we truncate the string field $|\Phi\rangle$ to lie in $\mathcal{H}^1_1$ by setting string fields in $\mathcal{H}^1_2$ to zero.
in this case the action is:

$$\tilde{S}(T) = V_{p+1}\tilde{\mathcal{L}(T)} = -V_{p+1}U(T)$$

where T is truncated string field and $\tilde{S}(T)$ is truncated action. action and potential are universal which means they made by ghost and Virasoro operators.

Mass of D-brane

consider a bosonic $D_p - brane$ with the following mode:

$$|\mathbf{\Phi}\rangle = \int \mathrm{d}^{\mathbf{p+1}}\mathbf{k}\phi^{\mathbf{i}}(\mathbf{k})\delta^{\mathbf{p}}(\overrightarrow{k})\mathbf{c_1}\alpha^{\mathbf{i}}_{\mathbf{-1}}\mathbf{e}^{\mathbf{ik.X(0)}}|\mathbf{0}\rangle$$

we evaluate quadric part of the action:

$$S_{quad} = -\frac{1}{2g_0^2\alpha'}\langle\Phi|Q_B|\Phi\rangle = -\frac{1}{2g_0^2\alpha'}\langle\mathcal{I}o\Phi(0)||(c_0L_0^{matter})\Phi(0)\rangle$$
$$= \frac{1}{4g_0^2\alpha'^2}\int \mathrm{d}^{p+1}\mathrm{d}^{q+1}\phi^i(k)\phi^j(q)(\frac{1}{\epsilon^2})^{\alpha' k^2}\delta^p(\overrightarrow{k})\delta^p(\overrightarrow{q})\langle e^{ikX}c\partial X^i(-\frac{-1}{\epsilon})|$$
$$\times (c_0L_0^m)|c\partial X^i e^{iqX}\rangle$$
$$= (2\pi)^{p+1}\frac{1}{2g_0^2}\int \mathrm{d}k_0\mathrm{d}q_0\phi^i(k_0)\phi^i(q_0)\delta^{ij}q_0^2\delta(k_0+q_0)\delta^p(\overrightarrow{0})(\frac{1}{\epsilon})^{-\alpha' k_0^2-\alpha' k_0q_0}$$
$$= \frac{\pi V_p}{g_0^2}\int \mathrm{d}k_0(k_0)^2\phi^i(k_0)\phi^i(-k_0)$$

in the last step, we have used $V_p = (2\pi)^p \delta^p(0)$. now by using Fourier transformation $\phi^i(k_0) = \int \frac{dt}{2\pi} \chi^i(t) e^{-ik_0 t}$, quadric part of action can be written as:

$$\mathbf{S_{quad}} = \frac{\mathbf{V_p}}{\mathbf{2g_0^2}} \int \mathrm{d}\mathbf{t} \partial_\mathbf{t} \chi^\mathbf{i} \partial_\mathbf{t} \chi^\mathbf{i}$$

this χ^i has the interpretation of the location of the $D_p - brane$ in the x^i direction. it's normalization is not correctly chosen. so we want to fix this.

We know from string theory: if we take a pair of identical D-branes whose locations are 0^i and b^i, and calculate the $(Mass)^2$ of open string we have:

$$M^2 \sim \frac{|\overrightarrow{b}|^2}{(2\pi\alpha')^2}$$

now if we move one of those d-branes (for example that one in b^i) by a small amount, $Y^i << 1$, the mass squared of open string changes into:

$$\Delta M^2 = \frac{|\overrightarrow{b} + \overrightarrow{Y}|^2}{(2\pi\alpha')^2} - \frac{|\overrightarrow{b}|^2}{(2\pi\alpha')^2} = \frac{\overrightarrow{b}.\overrightarrow{Y}}{(2\pi\alpha')^2} + O(\overrightarrow{Y^2})$$

or

$$\Delta M^2 \approx \frac{\overrightarrow{b}.\overrightarrow{Y}}{(2\pi\alpha')^2}$$

[4]

[4] A. Sen, "Universality of the tachyon potential", JHEP 12, 027 (2000), arXiv:hep-th/9911116,

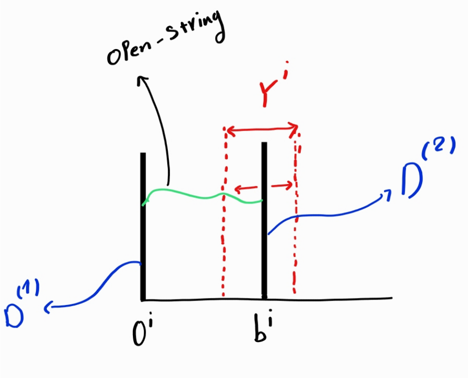

Figure: changing the position of $D^{(2)}$ by a small amount Y^i(red). open string stretched between $D^{(1)}$ and $D^{(2)}$(green).

Now we want to use SFT language to describe the situation: for simplicity consider tachyonic state with 2×2 Chan-Paton factors $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ as follow:

$$|T\rangle = \left(\int \mathrm{d}k_0 u(k_0) U_{k_0}(z=0) \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \int \mathrm{d}k_0 v(k_0) V_{k_0}(z=0) \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) |0\rangle$$

with these vertex operators:

$$U_{k_0}(z) = c e^{i \frac{b^i}{2\pi\alpha'} X^i} e^{i k_0 X^0}$$

$$V_{k_0}(z) = c e^{-i \frac{b^i}{2\pi\alpha'} X^i} e^{i k_0 X^0}$$

eigenvalue of L_0^{tot} for this state is $-\alpha'(k_0)^2 + \frac{\vec{b}^2}{(2\pi)^2} - 1$. we consider a string field background which has the effect of translation of one the two branes:

$$|tr\rangle = \chi^i P^i(z=0) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} |0\rangle \, ; \qquad P^i(z) = i \frac{1}{\sqrt{2\alpha'}} c \partial X^i(z)$$

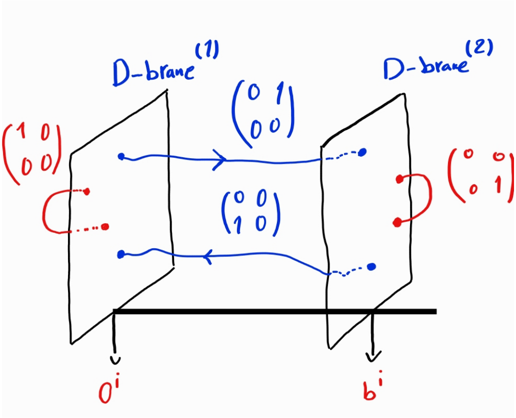

Figure: open strings (tachyonic state) stretched between $D^{(1)}$ and $D^{(2)}$(blue).

by plugging $|\Phi\rangle = |T\rangle + |tr\rangle$ into SFT action we have:

$$S = -\frac{1}{g_0^2}\left(\frac{1}{2\alpha'}\langle\Phi| Q_B |\Phi\rangle + \frac{1}{3}\langle\Phi| \Phi |\Phi\rangle\right)$$

after few steps and by considering quadric terms in u and v, we have:

$$S_{uv} = \frac{V_p}{g_0^2}\int \mathrm{d}t \partial_t u \partial_t v$$

for the cubic part of the action (only χuv terms) we have:

$$S_{\chi uv} = -\frac{V_p}{g_0^2}\left(\frac{1}{\pi\sqrt{2\alpha'}}\overrightarrow{\chi}.\overrightarrow{b}\right)\int \mathrm{d}tuv$$

by comparing these two parts of SFT action with scalar field's action we find:

$$\Delta M^2 = \frac{1}{\pi\sqrt{2\alpha'}}\overrightarrow{\chi}.\overrightarrow{b}$$

and finally comparing to the result from string theory, the normalization of χ is:

$$\chi^i = \frac{1}{\sqrt{2}\pi\alpha'^{\frac{3}{2}}}Y^i$$

now for quadric part of the action we can see:

$$S_{quad} = \frac{V_p}{4\pi^2 g_0^2 \alpha'^3} \int \mathrm{d}t \partial_t Y^i \partial_t Y^i$$

this contribution to the D-brane action can be interpreted as kinetic energy. so the tension of the $D_p - brane$ is:

$$\tau_p = \frac{1}{2\pi^2 g_0^2 \alpha'^3}$$

this tension may seem to include wrong dependence on α', but we should remember that g_0 is a dimensionful parameter. in order for S_{quad} to be dimensionless, we have:

$$[S_{quad}] = 0 \longrightarrow [g_0] = \frac{-p+5}{2}$$

by defining new, dimensionless coupling $\bar{g_0}^2 = g_0^2 \alpha'^{\frac{5-p}{2}}$, the correct formula is:

$$\tau_p = \frac{1}{2\pi^2 \bar{g_0}^2 \alpha'^{\frac{p+1}{2}}}$$

Sen's first conjecture and level truncation

First conjecture: The difference in potential between the unstable vacuum and stable one, should be the mas of $D_p - brane$.

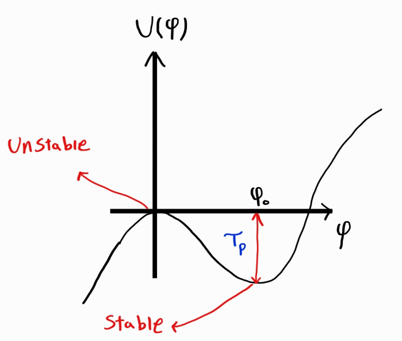

Figure: The tachyon potential. in unstable point string field is zero. at this point the Unstable D-brane is still present. According to the first conjecture the D-brane decays to stable vacuum which the difference in potential energy equals to τ_p.

Now we investigate evidence for the first conjecture by using level truncation. from previous results we can rewrite the action as:

$$\tilde{S}(T) = -\frac{1}{g_0^2}(\frac{1}{2\alpha'}\langle \mathcal{I}oT(0)Q_B T(0)\rangle + \frac{1}{3}\langle f_1 oT(0) f_2 oT(0) f_3 oT(0)\rangle)$$
$$= -(V_{p+1}2\pi^2\alpha'^3\tau_p)(\frac{1}{2\alpha'}\langle \mathcal{I}oT(0)Q_B T(0)\rangle + \frac{1}{3}\langle f_1 oT(0) f_2 oT(0) f_3 oT(0)\rangle)$$

where factor V_{p+1} comes from the fact that we are in Hilbert space $\mathcal{H}_1^1$ (or $T \in \mathcal{H}_1^1$) which in this case T state has no momentum, so we have:

$$(2\pi)^{p+1}\delta^{p+1}(\sum K) = (2\pi)^{p+1}\delta^{p+1}(0) = V_{p+1}$$

from our previous results about universality of tachyon potential we know $U(T) = \frac{-\tilde{S}(T)}{V_{p+1}}$. we define a new universal function as follow:

$$f(T) = \frac{U(T)}{\tau_p} = 2\pi^2\alpha'^3(\frac{1}{2\alpha'}\langle \mathcal{I}oT(0)Q_B T(0)\rangle + \frac{1}{3}\langle f_1 oT(0) f_2 oT(0) f_3 oT(0)\rangle)$$

sum of total energy and D-brane tension is given by:

$$U(T) + \tau_p = \tau_p(1 + f(T))$$

so if the first conjecture is correct we should have $f(T_0) = -1$, where T_0 is minimum point of tachyon potential. at first we pick level(0,0) truncation of the string field action with zero momentum tachyon state. the result is:

$$f(\phi) = 2\pi^2\alpha'^3(-\frac{1}{2\alpha'}\phi^2 + \frac{1}{3}(\frac{3\sqrt{3}}{4})^3\phi^3)$$

here we considered $\phi = \tilde{\phi}$ to have no derivative of ϕ. by solving $\frac{\partial f}{\partial \phi}|_{\phi_0} = 0$ we find:

$$\phi_0 = (\frac{4}{3\sqrt{3}})^3\frac{1}{\alpha'} \qquad \rightarrow \qquad f(\phi_0) \simeq -0.684$$

this is about 68 % of the value predicted by firs conjecture. if we proceed by considering higher level truncation we achieve more accurate results.

Level	$f(T_0)$
(0,0)	-0.684
(2,4)	-0.949
(2,6)	-0.959
(4,8)	-0.986
(4,12)	-0.988
(6,12)	-0.99514
(6,18)	-0.99518
(8,16)	-0.99777
(8,20)	-0.99793
(10,20)	-0.99912

Table: minimum values of tachyon potential at different level truncations.

Wedge states and KBc algebra

Let's remember conformal transformation on a primary field:

$$f o\phi(z) = (\frac{\mathrm{d}f}{\mathrm{d}z})^h \phi(f(z))$$

we can rewrite this equation using Virasoro generators:

$$f o\phi(z) = U_f \phi(z) U_f^{-1}, \qquad U_f = \exp(\sum_{n\geq 0} v_n L_n)$$

where we can determine v_n form the map f. To every map f we can define a state $\langle f|$, which is:

$$\langle f| \, |\phi\rangle = \langle f o\phi(0)\rangle_\Sigma$$

Where Σ is a surface (UHP or unit disk). we call these states ($\langle f|$) Surface state. they are related to U_f as:

$$\langle f| = \langle 0| \, U_f$$

[5]

[5] M. Kudrna, "Level Truncation Approach to Open String Field Theory," PhD thesis,[arXiv:2101.07678 [hep-th]].

A special subset of these surface states are called wedge state. wedge states are related to transformations:

$$f_r(z) = \tan(\frac{2}{r+1}\arctan(z))$$

These transformations are those $f_i(z_i) = h^{-1} o g_i(z_i)$ for integer r (but here r can be any arbitrary non-negative number). We denote wedge states as:

$$\langle r|_W \equiv \langle f_r| = \langle 0| U_{f_r}, \quad r \geq 0$$

we can construct $|r\rangle_W = U^*_{f_r}|0\rangle$, where $U^*_{f_r}$ is BPZ conjugate of U_{f_r}. The star product of wedge state is:

$$|r\rangle_W * |s\rangle_W = |r+s\rangle_W$$

so by this we can have:

$$|r\rangle_W = |0\rangle_W * |r\rangle_W$$

As you see, we can treat $|0\rangle_W$ as the identity element of wedge state algebra (more generally identity of whole star product algebra). we call this identity string field and denote as:

$$|I\rangle \equiv |0\rangle_W$$

this wedge state (identity string field) is associated with $f = \frac{2z}{1-z^2}$.

Also we have wedge state $|1\rangle_W$ as:

$$|1\rangle_W = |0\rangle$$

so we can construct any $|r\rangle_W$ by multiplying $|0\rangle$, r times:

$$|r\rangle_W = \underbrace{|0\rangle * \cdots * |0\rangle}_{r \text{ times}}$$

Wedge state also have an important property, which is:

$$Q_B |r\rangle_W = 0$$

This is because these states are made of total Virasoro operators (which commute with Q_B). we can visualize these states using silver frame, which defines with:

$$f_{silver} = \frac{2}{\pi} \arctan(z)$$

the wedge state $|r\rangle_W$ in this frame is a strip of width r, and star product by gluing of strips (also integration glues edges of strip to make cylinder).

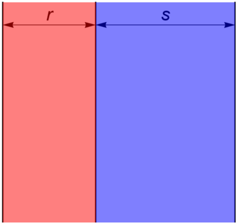

Figure: Star product of wedge states in silver frame.

$$|r\rangle_W * |s\rangle_W = |r+s\rangle_W$$

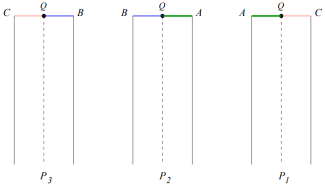

Figure: Representation of cubic vertex as gluing 3 strips.

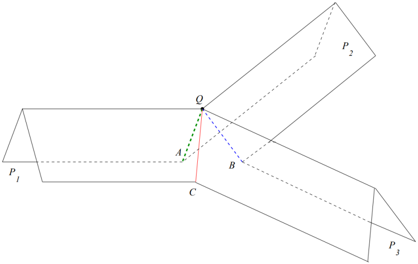

Figure: Result of gluing.

Wedge states can be rewrite using a string field K:

$$|r\rangle_W = e^{(-rK)}$$

which K is defined using line integral in silver frame:

$$K = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}\tilde{z}\tilde{T}(\tilde{z})\,|I\rangle$$

$\tilde{z}$ is silver frame coordinate and $\tilde{T}$ is total energy-momentum tensor in silver frame. Also, we define two other string fields B and c:

$$B = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}\tilde{z}\tilde{b}(\tilde{z})\,|I\rangle\,, \qquad c = \tilde{c}(\frac{1}{2})\,|I\rangle$$

these three string field have following (anti)commuting relations:

$$\begin{aligned}
&[K, B] = 0\\
&\{B, c\} = 1\\
&B^2 = c^2 = 0\\
&Q_B k = 0, \qquad Q_B B = K, \qquad Q_B c = cKc
\end{aligned}$$

These algebraic relations are known as KBc algebra.

Glance at Schnabl solution

It took almost twenty years from Witten's paper until Schnabl found the analytic solution of e.o.m. and proved Sen's conjectures. prior candidate solutions were numerical or too abstract and they did not pass the test for proving Sen's conjectures.
from the finite gauge transformation we have:

$$\Psi = e^{-\Lambda} Q e^{\Lambda}$$

after reparametrization of Λ and rewrite the above equation as:

$$\Psi = \Gamma^{-1}(\Lambda) Q \Gamma(\Lambda) \qquad \Gamma(\Lambda) = 1 + \Lambda + O(\Lambda^2)$$

Schnabl's solution is represented using:

$$\Gamma(\Lambda) = \frac{1}{1 - \lambda\Lambda}$$

where λ is a parameter, and the solution is a pure-gauge for $|\lambda| < 1$ and it is tachyon vacuum for $\lambda = 1$. for other measures of λ the solution does not converge. also Λ takes the form:

$$\Lambda = Bc(0) |0\rangle$$

the solution it self takes the form:

$$\Psi = (1 - \lambda\Lambda)Q\frac{1}{1 - \lambda\Lambda} = Q\Lambda\frac{\lambda}{1 - \lambda\Lambda}$$

we can also expand Ψ in powers of λ:

$$\Psi = \sum_{n=1}^{\infty} \lambda^n \Psi_n$$

after few steps we can write:

$$\Psi_n = (Q\Lambda)\Lambda^{n-1} = \frac{\mathrm{d}}{\mathrm{d}n}\psi_{n-1}$$

where:

$$\psi_n = \frac{2}{\pi} c_1 \left|0\right\rangle * B \left|n\right\rangle * c_1 \left|0\right\rangle$$

after using some tricks and Using these wedge states, Martin Schnabl suggested a solution in the form:

$$\Psi = \lim_{N\to\infty}(\sum_{n=0}^{N}\psi'_n - \psi_N)$$

where by prime we mean $\frac{\partial}{\partial n}$, and we have:

$$\psi_n = c_1 |0\rangle * B |n\rangle * c_1 |0\rangle$$

for example we can check for $n = 0$, we have:

$$\psi_0 = (cBc)(0) |0\rangle , \quad \psi'_0 = (cBKc)(0) |0\rangle$$

from e.o.m. of SFT action we get(using BPZ inner product):

$$\langle \Psi, Q_B \Psi \rangle = - \langle \Psi, \Psi * \Psi \rangle$$

Schnabl showed from explicit form of solution we have:

$$\langle \Psi, Q_B \Psi \rangle = -\frac{3}{\pi^2}$$

so by using these we have:

$$-\frac{S}{V_{p+1}} = -\frac{1}{2\pi^2 g_0^2}$$

then first conjecture has been proved.

summary

- Introduced Open SFT and Witten's vertex;
- checked some simple calculations in level truncation and BCFT;
- checked universality of tachyon potential;
- checked evidence of Sen's first conjecture;
- Short review on analytic solution.