# Exercise 1 — Lorentz Symmetry

## 1. Lorentz invariance

- Consider the Klein-Gordon equation for the relativistic wave function $\psi(x)$ and show that if $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, this equation is invariant providing that

$$\psi'(x') = \psi(x)\,. \tag{1}$$

- Try to ask the same question in the Dirac equation this time with the condition

$$\psi'(x') = S(\Lambda)\psi(x) \tag{2}$$

  where $S(\Lambda)$ is a $4 \times 4$ matrix depending on $\Lambda$ and show that $S(\Lambda)^{-1}\gamma^\beta S(\Lambda) = \Lambda^\beta{}_\alpha \gamma^\alpha$.

- Now consider three coordinate systems $x$, $x' = \Lambda x$ and $x'' = \Lambda' x'$ by applying the same condition and the fact that the Dirac equation should remain invariant, show that $S(\Lambda)$ is a four dimensional representation of the Lorentz group called spinor representation.

## 2. Lorentz group

- In the neighbourhood of the identity $I$, a Lorentz transformation $\Lambda$ can be written as

$$\Lambda = I + \omega\,, \tag{3}$$

  by inserting this expression into the definition of a Lorentz transformation $\Lambda^T g \Lambda = g$ (in the component form $g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}$) show that $\omega$ is antisymmetric.

- Consider the **vector** representation of the Lorentz group $D(\Lambda)$ as

$$D(\Lambda) = \exp\left(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) \tag{4}$$

  where $M_{\mu\nu}$ are the antisymmetric $4 \times 4$ matrices and the the factor $i$ is chosen such that they are hermitian. Consider the infinitesimal transformation of $x^\mu$ once with $\Lambda$ and once with $D(\Lambda)$ as $\delta x^\mu = x'^\mu - x^\mu$. Show that

$$\omega^\mu{}_\nu = \frac{i}{2}\omega^{\rho\sigma}(M_{\rho\sigma})^\mu{}_\nu. \tag{5}$$

Conclude that

$$(M_{\rho\sigma})^{\mu}{}_{\nu} = i\left(g_{\rho\nu}\delta^{\mu}_{\sigma} - g_{\sigma\nu}\delta^{\mu}_{\rho}\right)\,. \tag{6}$$

- Show that the matrices $M_{\rho\sigma}$ satisfy the following commutation relation

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(g_{\mu\rho}M_{\nu\sigma} - (\mu \leftrightarrow)) - (\rho \leftrightarrow \sigma)\,. \tag{7}$$

  These commutation relations specify the Lie algebra of the Lorentz group.

- Show that the generators of spatial rotation (angular momentum) and the boost can be identified as

$$J_i = \epsilon_{ijk}M^{jk}\,, \quad \text{and} \quad K_i = M^{i0}\,. \tag{8}$$

  rewrite the commutators in terms of $J_i$ and $K_i$.

- Define an new basis $S_i$ and $T_i$ in terms of $J_i$ and $K_i$ such that

$$[S_i, S_j] = i\epsilon_{ijk}S_k \quad [T_i, T_j] = i\epsilon_{ijk}T_k \quad [S_i, T_j] = 0\,. \tag{9}$$

  This means that $S_i$ and $T_i$ satisfy the commutation relation of the Lie algebra su(2).

3. **Poincaré group.** This group consists of the elements $(\Lambda, a)$.

- Show that all transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$ leave the distance squared $(x-y)^2$ invariant.

- By transforming the point $x$ two times under this symmetry group, show that they follow the composition law

$$(\Lambda_2, a_2)\cdot(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)\,. \tag{10}$$

- Find the identity element and the inverse element of the Poincaré group.

- consider the representation of the Poinacaré group as $D(\Lambda, a)$ with the infinitesimal form

$$D(\Lambda, a) = I + \frac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma} + ia_\mu P^\mu\,, \tag{11}$$

where $M_{\mu\nu}$ and $P_\mu$ are generators of Lorentz transformation and translations. Show that

$$D(\Lambda,0)^{-1}M^{\mu\nu}D(\Lambda,0) = \Lambda^{\mu\nu}{}_{\rho\sigma}M^{\rho\sigma} \tag{12}$$

$$D(\Lambda,0)^{-1}P^{\mu}D(\Lambda,0) = \Lambda^{\mu}{}_{\rho}P^{\rho}\,. \tag{13}$$

- Derive the commutation relations among $M_{\mu\nu}$ and $P_\mu$ which is the Poinacaré algebra.