Exercise 2— Fermions

1. **Spinor representation**

- Consider 4×4 anti-sym. matrices $S^{\rho\sigma}$ defined as

$$S^{\rho\sigma} = \frac{i}{2}\gamma^\rho\gamma^\sigma, \quad \rho \neq \sigma \tag{1}$$

where γ^ρ matrices satisfy the Clifford algebra:

$$\{\gamma^\rho, \gamma^\sigma\} = -2\eta^{\rho\sigma} I_{4\times 4} \tag{2}$$

show that $S^{\rho\sigma}$ satisfy the Lorentz algebra. (Hint: first calculate $[S^{\mu\nu}, \gamma^\rho]$.)

- Introduce the finite rep.

$$S(\Lambda) = \exp\left(\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right) \tag{3}$$

where $\omega_{\rho\sigma}$ are six real numbers as introduced in $D(\Lambda)$ before. Now construct $S(\Lambda)$ for rotations and for boosts using S_{ij} and S_{0i} (the same as you did with M_{ij} and M_{0i} before.) Hint: use the rep. :

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{4}$$

were σ^i denotes Pauli matrices. We can write this representation more compactly by introducing $\sigma^\mu = (I, \vec{\sigma})$ and $\bar{\sigma}^\mu = (I, -\vec{\sigma})$ as

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{5}$$

This is an irreducible representation of the Clifford algebra called Weyl or chiral representation. For any invertible matrix V one can construct an equivalent representation of the Clifford algebra $V\gamma^\mu V^{-1}$.

- Show that

$$S(\Lambda_{\text{rot}}) = \begin{pmatrix} e^{i\vec{\phi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\phi}\cdot\vec{\sigma}/2} \end{pmatrix} \tag{6}$$

for rotation and

$$S(\Lambda_{\text{boost}}) = \begin{pmatrix} e^{\vec{\chi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\chi}\cdot\vec{\sigma}/2} \end{pmatrix} \tag{7}$$

for boost.

- Show that $S(\Lambda_{\text{rot}}) = -1$ for 2π-rotation along x^3-axis.
- Show that $S(\Lambda_{\text{boost}})^\dagger S(\Lambda_{\text{boost}}) \neq 1$ (the boost rep. is non-unitary).
- Show that $S(\Lambda_{\text{boost}})^\dagger = \gamma^0 S(\Lambda_{\text{boost}})^{-1}\gamma^0$.

This shows that the spinor representation is non-unitary. In fact there are no finite dimensional unitary representation of the Lorentz group.

2. Dirac equation

- Show that the Dirac action

$$S = \int \mathrm{d}^4x \bar{\Psi}(x)(i\gamma^\mu\partial_\mu - m)\Psi(x) \tag{8}$$

is Lorentz inv. (scalar) and gives the Dirac eq. as the Euler-Lagrange eq.

$$(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0\,. \tag{9}$$

- Show that the plane-wave ansatz

$$\Psi(\vec{x}) = u(\vec{p})e^{-ipx}, \qquad \Psi(\vec{x}) = v(\vec{p})e^{+ipx} \tag{10}$$

are positive and negative frequency solutions to the Dirac eq. iff:

$$(\not{p} + m)u(\vec{p}) = 0, \qquad (-\not{p} + m)v(\vec{p}) = 0 \tag{11}$$

For $\vec{p} = 0$ (which means $\not{p} = -m\gamma^0$) show that these equations can be solved as;

$$u(\vec{0}) = \begin{pmatrix} a \\ b \\ a \\ b \end{pmatrix}, \qquad v(\vec{0}) = \begin{pmatrix} c \\ d \\ -c \\ -d \end{pmatrix} \tag{12}$$

So the two linearly independent solutions are:

$$u^{(1)}(\vec{0}) = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \qquad u^{(2)}(\vec{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix} \tag{13}$$

$$v^{(1)}(\vec{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \qquad v^{(2)}(\vec{0}) = \sqrt{m}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix} \tag{14}$$

- Use $\bar{\Psi} = \Psi^\dagger \gamma^0$ to show that

$$\bar{u}^{(1)}(\vec{0}) = \sqrt{m}(1,0,1,0), \quad \bar{u}^{(2)}(\vec{0}) = \sqrt{m}(0,1,0,1) \tag{15}$$

$$\bar{v}^{(1)}(\vec{0}) = \sqrt{m}(0,-1,0,1), \quad \bar{v}^{(2)}(\vec{0}) = \sqrt{m}(1,0,-1,0) \tag{16}$$

- Now construct the spin operator

$$S_z = \frac{i}{4}\left[\gamma^1,\gamma^2\right] = \frac{i}{2}\gamma^1\gamma^2 = \frac{1}{2}\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix} \tag{17}$$

and show that for $s = 1,2$:

$$S_z u^{(s)}(0) = \pm\frac{1}{2}u^{(s)}(0)\,, \tag{18}$$

$$S_z v^{(s)}(0) = \mp\frac{1}{2}v^{(s)}(0)\,. \tag{19}$$

- Use the boost generators $\vec{K}$ (in spinor rep.) for going to arbitrary 3-momentum $\vec{p} = (0,0,p_z)$:

$$u^{(s)}(\vec{p}) = \exp(i\eta K^3)u^{(s)}(0),\, , \qquad v^{(s)}(\vec{p}) = \exp(i\eta K^3)v^{(s)}(0) \tag{20}$$

Do the same for $\bar{u}^{(s)}(\vec{p})$ and $\bar{v}^{(s)}(\vec{p})$. What is the value of η (rapidity)? (Hint: use the boost matrix K^3 for acting on the 4-vector $(m,\vec{0})$. Answer: $\eta = \sinh^{-1}(\frac{p_z}{m})$.)